Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 41 (2004) 7329–7349
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Nonlocal and gradient rate plasticity

Francesco Marotti de Sciarra *

Dipartimento di Scienza delle Costruzioni, Facolta' di Ingegneria, Universita' degli Studi di Napoli Federico II, 80125 Napoli, Italy

Received 11 May 2004
Available online 31 July 2004

This paper is dedicated to Angela on the occasion of her 8th birthday

Abstract

This paper deals with a formulation of nonlocal and gradient plasticity with internal variables. The constitutive model complies with local internal variables which govern kinematic hardening and isotropic softening and with a nonlocal corrective internal variable defined either as the sum between a new internal variable and its spatial weighted average or as the gradient of a measure of plastic strain. The rate constitutive problem is cast in the framework provided by the convex analysis and the potential theory for monotone multivalued operators which provide the suitable tools to perform a theoretical analysis of such nonlocal and gradient problems. The validity of the maximum dissipation theorem is assessed and constitutive variational formulations of the rate model are provided. The structural rate problem for an assigned load rate is then formulated. The related variational formulation in the complete set of state variable is contributed and the methodology to derive variational formulations, with different combinations of the state variables, is explicitly provided. In particular the generalization to the present nonlocal and gradient model of the principles of Prager–Hodge, Greenberg and Capurso–Maier is presented. Finally nonlocal variational formulations provided in the literature are derived as special cases of the proposed model.

Keywords: Nonlocal plasticity; Gradient plasticity; Internal variables; Strain localization; Variational formulations

1. Introduction

The increasing interest in generalizing continuum models of plasticity and damage has its origin in the serious drawbacks of these classical theories when a strain softening behaviour is exhibited. In fact most engineering materials, such as metals, concrete, fiber-reinforced materials and soils, show a loss of positive definiteness of the tangent stiffness operator which yields to the localization of plastic deformations and of damage in narrow bands until the occurrence of cracks appear.

The deformation pattern in a body in which a localization phenomenon occurs suddenly evolves from relatively smooth into one in which shear bands of highly strained material appears whereas the remainder

* Tel.: +39-081-7683336; fax: +39-081-7683332.
E-mail address: marotti@unina.it (F. Marotti de Sciarra).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.05.026

part of the body unloads. In classical theories of plasticity and damage the size of the localization zone is unspecified due to the lack of any material internal length and, if the material does not have a residual strength, the energy dissipated in the localized zone tends to zero as the mesh is refined (see e.g. Lasry and Belytschko, 1988). Accordingly in a finite element analysis, the solution depends on the mesh size employed since the localization band tends to the smallest finite element.

On the contrary, the presence of a length scale in plasticity theories reflects the ability of the microstructure to transmit information to neighbouring points within a certain distance and turns out to be a material property linking the microstructure to the continuum. There are different theories in literature which introduce a length scale in a continuum model in order to bypass the inadequacy of standard rate-independent continuum models to deal with the problem of strain localization. One way is to consider viscoplastic models (Needleman, 1988; Sluys and de Borst, 1992) or the Cosserat theory (Cosserat and Cosserat, 1909; Ericksen and Truesdell, 1958). Other possibilities involve gradient of strains or of plastic variables or are based on nonlocal (integral) theories.

The nonlocal theory has been proposed, at first, by Eringen (1966) and Eringen and Edelen (1972) and then, for concrete and damaging materials, see e.g. Bažant et al. (1984) and Pijaudier-Cabot and Bažant (1987). The key idea is to introduce, in the constitutive model, state variables defined in an average form over a finite volume of the body (de Borst, 2001) and the material length parameter determines how the value of the variable at a certain point is weighted (Bažant and Pijaudier-Cabot, 1988; Pijaudier-Cabot and Benallal, 1993). The gradient plasticity is a variant of the nonlocal approach in which higher-order spatial derivatives are introduced in the constitutive equations. It was first suggested by Dillon and Kratochvil (1970), which motivated the consideration of strain gradient as one way to account for the interaction among dislocation, and then by Aifantis (1987) and Coleman and Hodgdon (1985) for rigid plastic materials and, subsequently, it has been developed in several papers, see e.g. Mühlhaus and Aifantis (1991), de Borst and Mühlhaus (1992), de Borst et al. (1993, 1995) and de Borst (2001).

We consider here materials such that microstructural effects become significant only at the onset of localization. Accordingly we assume that the classical continuum elastoplastic theory (without any nonlocal effect) satisfactorily predicts the orientation of the localized deformation band.

In modern plasticity theory, the inelastic constitutive behaviour is described in terms of internal variables. In the nonlocal approach, the form of the constitutive relations is left unchanged and the yield condition is modified by introducing a nonlocal internal variable. In this view, a thermodynamic formulation of a nonlocal plastic model has been presented in Borino et al. (1999) in the hypothesis that the inelastic behaviour is governed by a nonlocal internal variable describing isotropic hardening. In fact a computational effective model has been proposed in Svedberg and Runesson (1998) in which the nonlocal internal variable governing isotropic hardening is obtained as the difference between a nonlocal field and a local field.

The aim of this paper is to cast nonlocal and gradient plasticity models with internal variables in the framework of the convex analysis which provides the appropriate tools for a theoretical analysis of such problems. It is also shown how it is possible to consistently derive nonlocal variational formulations to be used for a subsequent finite element analysis.

In the present paper a unified constitutive model of nonlocal and gradient standard plasticity is set in the framework of the generalized standard material introduced by Halphen and Nguyen (1975) and Nguyen (1977). The free energy and the elastic domain are respectively defined in the product space of kinematic and static generalized variables. The nonlocal and gradient behaviour is obtained by a suitable definition of the free energy and a nonlocal version of the maximum dissipation theorem is proved. Then the response of the material to a given total strain rate is developed and the related variational formulations are provided by appealing to the properties of saddle functionals and of local subdifferentiability.

The structural rate problem is then addressed. An original variational formulation in the complete set of local and nonlocal state variables is provided and the systematic derivation of variational formulations with different combinations of state variables is presented. In particular the generalization to the nonlocal and gradient context of the classical variational principles of Prager and Hodge (1951), Greenberg (1949), Capurso (1969), and Capurso and Maier (1970) involving the plastic multiplier, are provided.

Moreover, it is shown that the nonlocal model here proposed encompasses, as a special case, the constitutive nonlocal (integral) models provided in Borino et al. (1999) and Borino and Failla (2001).

The variational formulations contributed in Borino et al. (1999) are recovered by a suitable specialization of the variational formulations proposed in this paper. A critical comparison between these variational formulations shows that the nonnegativeness of a quadratic form involving plastic multipliers, appearing in a variational principle reported in the previously quoted paper, can be omitted by appealing to the property of local convexity.

Finally the variational principle for gradient plasticity proposed by Mühlhaus and Aifantis (1991), and referred to in de Borst and Mühlhaus (1992), Fleck and Hutchinson (2001) among others, is recovered as a special case of the proposed formulation.

2. Nonlocal and gradient plasticity

We analyse a nonlocal elastoplastic structural problem defined on a regular bounded domain Ω of an Euclidean space. The inelastic model is cast in the framework of internal variable theories of associated type and the *generalized standard material* (Halphen and Nguyen, 1975) is considered.

The dual spaces of strains ε and stresses σ will be labelled by $\mathscr{D}$ and $\mathscr{S}$ respectively and the elastic strain will be denoted by $e \in \mathscr{D}$. The internal variables account for the evolution of the hardening/softening phenomena. Following the approach proposed by Halphen and Nguyen (1975), the kinematic (strain-like) internal variables are denoted by $\kappa \in \mathscr{Y}$, $\alpha_1 \in \mathscr{Y}_1$, $\alpha_2 \in \mathscr{Y}_2$ and the dual (stress-like) static internal variables are $X \in \mathscr{Y}'$, $\chi_1 \in \mathscr{Y}'_1$, $\chi_2 \in \mathscr{Y}'_2$. From a mechanical point of view, the static internal variable χ_1 describes the kinematic hardening, the variable χ_2 describes the nonlocal isotropic hardening and X describes the isotropic softening behaviour.

It will be shown in the sequel that the rates of $-\kappa$ and α_2 coincide with the plastic multiplier and the rate of α_1 has the direction of the inward normal to the elastic domain. The symbol $((\bullet,\bullet))$ denotes the inner product in the dual spaces and has the mechanical meaning of the internal virtual work. For the Cauchy model we have

$$((\bullet,\bullet)) = \int_\Omega \bullet \cdot \bullet \, \mathrm{d}\Omega,$$

where $\cdot$ denotes the simple (double) index saturation operation between vectors (tensors).

In order to properly define the constitutive model, it is necessary to introduce the saddle (convex–concave) differentiable functional $\Phi : \mathscr{D} \times \mathscr{Y} \times \mathscr{Y}_1 \times \mathscr{Y}_2 \to \overline{\Re}$ (where $\overline{\Re} = \{-\infty \cup \Re \cup +\infty\}$) representing the *free energy* and the convex elastic domain C in the product space $S \times \mathscr{Y}' \times \mathscr{Y}'_1 \times \mathscr{Y}'_2$.

The free energy is additively decomposed in the sum of a strictly convex potential $\Phi_{\mathrm{el}}(e)$, representing the elastic energy, and of a saddle functional $\Phi_{\mathrm{in}}(\alpha_1,\alpha_2,\kappa)$, convex in (α_1,α_2) and concave in κ, which accounts for inelastic phenomena. Such a decomposition corresponds to the mechanical assumption that the elastic behaviour does not depend on the evolution of inelastic phenomena and has been usually adopted in literature concerning local and gradient plasticity, see e.g. Lubliner (1990) Reddy and Martin (1991), Simo et al. (1988), Strömber and Ristinmaa (1996), and Svedberg and Runesson (1998).

Nonlocal effects can then be modelled by giving the following expression to the free energy:

$$\Phi(e,\alpha_1,\alpha_2,\kappa) = \Phi_{\mathrm{el}}(e) + \Phi_{\mathrm{in}}(\alpha_1,\alpha_2,\kappa) = \Phi_{\mathrm{el}}(e) + \Phi_L(\alpha_1,\kappa) + \Phi_{NL}(\alpha_2), \tag{1}$$

where the free energy component $\Phi_L(\alpha_1, \kappa)$ is saddle in (α_1, κ) and $\Phi_{NL}(\alpha_2) = \Phi_{NL}(\xi(\alpha_2))$ is the convex nonlocal part of the free energy.

A nonlocal plastic behaviour can be modelled by assuming that the functional Φ_{NL} at a point $\mathbf{x}$ of the body Ω depends on the entire field α_2. This task can be achieved by considering the nonlocal kinematic (strain-like) variable $\xi \in \mathscr{Z}$ having the following parametric representation:

$$\xi(\mathbf{x}) = (\mathbf{R}\alpha_2)(\mathbf{x}), \tag{2}$$

where $\mathbf{R} : \mathscr{Y}_2 \to \mathscr{Z}$ denotes a suitable linear regularization operator (Pijaudier-Cabot and Bažant, 1987; Strömber and Ristinmaa, 1996; Borino et al., 1999). The kinematic internal variable ξ turns out to be nonlocal since its value at the point $\mathbf{x}$ of the body Ω depends on the entire field α_2. Since a nonlocal behaviour must be present for high space variation of the local variable α_2, we assume that the kernel of $\mathbf{R}$ is provided by uniform fields.

The expression (2) encompasses both nonlocal and gradient plasticity as shown hereafter.

- *Nonlocal plasticity*: A nonlocal kinematic field ξ can be obtained as a spatial weighted average of the variable α_2 in the form:

$$\xi(\mathbf{x}) = (\mathbf{R}\alpha_2)(\mathbf{x}) = \frac{1}{V(\mathbf{x})} \int_\Omega \beta_{\mathbf{x}}(\mathbf{y})\alpha_2(\mathbf{y})\,\mathrm{d}\mathbf{y}, \qquad V(\mathbf{x}) = \int_\Omega \beta_{\mathbf{x}}(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{3}$$

where $\beta_{\mathbf{x}}(\mathbf{y})$ is a spatial weighting function depending on a material parameter called the internal length scale and $V(\mathbf{x})$ is the representative volume at the point $\mathbf{x}$.

From a computational point of view (see e.g. Svedberg and Runesson, 1998; Borino and Failla, 2001) it is convenient to perform an additive decomposition of the nonlocal internal variable ξ in the form

$$\xi(\mathbf{x}) = \frac{1}{V(\mathbf{x})} \int_\Omega \gamma_{\mathbf{x}}(\mathbf{y})\alpha_2(\mathbf{y})\,\mathrm{d}\mathbf{y} - \alpha_2(\mathbf{y}), \tag{4}$$

where $\gamma_{\mathbf{x}}(\mathbf{y})$ is a spatial weighting function. The first term at the r.h.s. of (4) complies with a nonlocal behaviour and the second term complies with a local behaviour. Denoting by $\Delta_{\mathbf{x}}(\mathbf{y})$ the Dirac delta centred at the point $\mathbf{x}$, the expression (4) can be modified in the form:

$$\xi(\mathbf{x}) = \frac{1}{V(\mathbf{x})} \int_\Omega \gamma_{\mathbf{x}}(\mathbf{y})\alpha_2(\mathbf{y})\,\mathrm{d}\mathbf{y} - \int_\Omega \Delta_{\mathbf{x}}(\mathbf{y})\alpha_2(\mathbf{y})\,\mathrm{d}\mathbf{y}, \tag{5}$$

and it turns out to be apparent that the expression (5) is of the same form of relation (3) by assuming $\beta_{\mathbf{x}}(\mathbf{y}) = \gamma_{\mathbf{x}}(\mathbf{y}) - V(\mathbf{x})\Delta_{\mathbf{x}}(\mathbf{y})$. Accordingly the nonlocal form (4) can be viewed as a special case of (3) which will be referred to in the sequel.

- *Gradient plasticity*: The regularization operator is chosen as a differential operator (see e.g. Mühlhaus and Aifantis, 1991; de Borst and Mühlhaus, 1992; de Borst, 2001) so that $\mathbf{R}$ assumes the following form:

$$\xi(\mathbf{x}) = (\mathbf{R}\alpha_2)(\mathbf{x}) = c(\nabla\alpha_2)(\mathbf{x}), \tag{6}$$

  where ∇ denotes the gradient operator and c is a length parameters which tends to 0 for $V \to 0$.

Accordingly the expression (2) provides in a unitary framework a nonlocal variable or a gradient one so that, in the sequel, we will refer to the generic form of the nonlocal variable (2).

Let us now provide the constitutive relations for nonlocal and gradient plasticity. Recalling the expression (2) of the nonlocal kinematic variable ξ, the constitutive relations can be obtained from the saddle free energy (1) as follows:

$$(\sigma,\chi_1,\chi_2,-X)=\mathrm{d}\Phi(e,\alpha_1,\alpha_2,\kappa)\iff\begin{cases}\sigma=\mathrm{d}\Phi_{\mathrm{el}}(e),\\ \chi_1=\mathrm{d}_{\alpha_1}\Phi_L(\alpha_1,\kappa),\\ \chi_2=\mathrm{d}_{\alpha_2}\Phi_{NL}(\xi(\alpha_2))=\mathbf{R}'\,\mathrm{d}\Phi_{NL}(\xi)=\mathbf{R}'\chi,\\ -X=\mathrm{d}_\kappa\Phi_L(\alpha_1,\kappa),\end{cases}\tag{7}$$

where we have set $\chi=\mathrm{d}\Phi_{NL}(\xi)\in\mathscr{X}'$ and $\mathbf{R}':\mathscr{X}'\to\mathscr{Y}_2'$ denotes the dual operator of $\mathbf{R}$.

Note that the static internal variable χ_2, which is dual of the (local) kinematic internal variable α_2, is a *global* variable since its pointwise value depends upon the entire field χ over the body Ω, i.e. $\chi_2(\mathbf{x})=(\mathbf{R}'\chi)(\mathbf{x})$ for any $\mathbf{x}\in\Omega$.

Starting from (3), it is immediate to prove that the dual regularization operator for nonlocal plasticity is given by

$$(\mathbf{R}'\chi)(\mathbf{x})=\int_\Omega\frac{1}{V(\mathbf{y})}\beta_{\mathbf{y}}(\mathbf{x})\chi(\mathbf{y})\,\mathrm{d}\mathbf{y}.$$

The operator $\mathbf{R}$ is self-adjoint, i.e. $\mathbf{R}'=\mathbf{R}$, if the representative volume V does not depend on the point $\mathbf{x}$.

If a linear nonlocal hardening behaviour is assumed, the expression of Φ_{NL} is given by

$$\Phi_{NL}(\alpha_2)=\frac{1}{2}((h\xi(\alpha_2),\xi(\alpha_2)))=\frac{1}{2}((h\mathbf{R}\alpha_2,\mathbf{R}\alpha_2))=\frac{1}{2}\int_\Omega\frac{h}{V(\mathbf{x})^2}\left[\int_\Omega\beta_{\mathbf{y}}(\mathbf{x})\alpha_2(\mathbf{y})\,\mathrm{d}\mathbf{y}\right]^2\mathrm{d}\mathbf{x},\tag{8}$$

where $h:\mathscr{X}\to\mathscr{X}'$ is a positive hardening modulus so that we have $\chi_2=\mathbf{R}'h\xi(\alpha_2)=(\mathbf{R}'h\mathbf{R})\alpha_2$.

In the case of gradient plasticity we have

$$\Phi_{NL}(\alpha_2)=\frac{1}{2}((h\xi(\alpha_2),\xi(\alpha_2)))=\frac{1}{2}((h\mathbf{R}\alpha_2,\mathbf{R}\alpha_2))=\frac{1}{2}\int_\Omega hc^2\nabla\alpha_2\cdot\nabla\alpha_2\,\mathrm{d}\mathbf{x},\tag{9}$$

and the duality between $\chi_2=\mathbf{R}'\chi$ and α_2 yields

$$((\chi,\mathbf{R}\alpha_2))=((\mathbf{R}'\chi,\alpha_2))\quad\forall\alpha_2\in\mathscr{Y}_2,\chi\in\mathscr{X}'$$

which provides the Green's formula:

$$\int_\Omega\chi\cdot\nabla\alpha_2\,\mathrm{d}\mathbf{x}=-\int_\Omega(\operatorname{div}\chi)\alpha_2\,\mathrm{d}\mathbf{x}+\int_{\partial\Omega}\chi\cdot\boldsymbol{n}(\boldsymbol{\Gamma}\alpha_2)\,\mathrm{d}\mathbf{s},$$

where $\boldsymbol{n}$ is the outward normal at the boundary $\partial\Omega$ of Ω and $\boldsymbol{\Gamma}\alpha_2$ yields the values of α_2 at the boundary of Ω, i.e. $\boldsymbol{\Gamma}\alpha_2=\alpha_2|_{\partial\Omega}$.

The constitutive model is completed by introducing the elastic domain C which is defined in the space of stresses and of static internal variables (σ,χ_1,χ_2,X) as the level set of a convex *yield mode* $G:\mathscr{S}\times\mathscr{Y}_1'\times\mathscr{Y}_2'\times\mathscr{Y}'\to\Re\cup\{+\infty\}$ in the form:

$$C=\{(\sigma,\chi_1,\chi_2,X)\in\mathscr{S}\times\mathscr{Y}_1'\times\mathscr{Y}_2'\times\mathscr{Y}':G(\sigma,\chi_1,\chi_2,X)\leqslant 0\}\tag{10}$$

provided that the minimum of G is negative.

The constitutive model can be formulated in a more convenient way by defining the following generalized vectors collecting together local and nonlocal variables:

$$\boldsymbol{\varepsilon}=\begin{bmatrix}\varepsilon\\0\\0\end{bmatrix},\quad\boldsymbol{e}=\begin{bmatrix}e\\\alpha_1\\\alpha_2\end{bmatrix},\quad\boldsymbol{p}=\begin{bmatrix}p\\-\alpha_1\\-\alpha_2\end{bmatrix},\quad\boldsymbol{\sigma}=\begin{bmatrix}\sigma\\\chi_1\\\chi_2\end{bmatrix}.\tag{11}$$

The vectors $\boldsymbol{\varepsilon}$, $\boldsymbol{e}$, $\boldsymbol{p}$ and $\boldsymbol{\sigma}$ represent the generalized vectors of total strain, elastic strain, plastic strain and stress. Accordingly two generalized spaces $\widehat{\mathscr{D}}=\mathscr{D}\times\mathscr{Y}_1\times\mathscr{Y}_2$ and $\widehat{\mathscr{S}}=\mathscr{S}\times\mathscr{Y}_1'\times\mathscr{Y}_2'$ denotes the generalized spaces of strains and of stresses and the scalar product between generalized vectors is

$((\boldsymbol{\sigma},\boldsymbol{e})) = ((\sigma,e)) + ((\chi_1,\alpha_1)) + ((\chi_2,\alpha_2))$. In the sequel, for simplicity, the term generalized will be omitted since no confusion can arise.

3. The elastic domain

In the applications the yield mode, defining the elastic domain C, is usually written in the form:

$$G(\boldsymbol{\sigma},X) = G(\sigma,\chi_1,\chi_2,X) = g(\sigma,\chi_1) - \chi_2 - X - \sigma_0, \tag{12}$$

where g is a convex function and σ_0 represents a constant scalar value which characterize the initial yield limit. The choice of the function g depends on the particular yield criterion adopted for the material, see e.g. Salençon (1983).

The flow rule can be formulated in terms of the normal cone N_C to the elastic domain C as follows:

$$(\dot{\boldsymbol{p}},\dot{\kappa}) = N_C(\boldsymbol{\sigma},X) = \partial \sqcup_C (\boldsymbol{\sigma},X) \iff (\dot{p},-\dot{\alpha}_1,-\dot{\alpha}_2,\dot{\kappa}) = N_C(\sigma,\chi_1,\chi_2,X), \tag{13}$$

where $\sqcup_C(\boldsymbol{\sigma},X)$ is the indicator of the elastic domain and turns out to be zero if $(\boldsymbol{\sigma},X) \in C$ and $+\infty$ otherwise (see Appendix C). The static internal variable X and its dual kinematic internal variable κ are kept separate from $\boldsymbol{\sigma}$ and $\boldsymbol{p}$ respectively to point out that they are associated with a softening behaviour.

The flow rule (13) can then be reformulated in the following three equivalent forms:

$$\begin{aligned} &(\dot{\boldsymbol{p}},\dot{\kappa}) \in N_C(\boldsymbol{\sigma},X), \qquad (\boldsymbol{\sigma},X) \in \partial D(\dot{\boldsymbol{p}},\dot{\kappa}),\\ &\sqcup_C (\boldsymbol{\sigma},X) + D(\dot{\boldsymbol{p}},\dot{\kappa}) = ((\boldsymbol{\sigma},\dot{\boldsymbol{p}})) + ((X,\dot{\kappa})), \end{aligned} \tag{14}$$

where $D : \widehat{\mathscr{D}} \times \mathscr{Y} \to \Re \cup \{+\infty\}$ is the plastic dissipation associated with the plastic flow $(\dot{\boldsymbol{p}},\dot{\kappa})$ whose expression is given by the support functional of the elastic domain C:

$$\begin{aligned} D(\dot{\boldsymbol{p}},\dot{\kappa}) &= \sup\{((\bar{\boldsymbol{\sigma}},\dot{\boldsymbol{p}})) + ((\bar{X},\dot{\kappa}))|(\bar{\boldsymbol{\sigma}},\bar{X}) \in C\}\\ &= \sup\{((\bar{\sigma},\dot{p})) - ((\overline{\chi}_1,\dot{\alpha}_1)) - ((\overline{\chi}_2,\dot{\alpha}_2)) + ((\overline{X},\dot{\kappa}))|(\bar{\sigma},\overline{\chi}_1,\overline{\chi}_2,\overline{X}) \in C\}. \end{aligned} \tag{15}$$

The sup operation is performed with respect to the over barred state variables $\bar{\bullet}$. Explicitly the relations (14) are

$$\begin{cases} (\dot{p},-\dot{\alpha}_1,-\dot{\alpha}_2,\dot{\kappa}) \in N_C(\sigma,\chi_1,\chi_2,X),\\ (\sigma,\chi_1,\chi_2,X) \in \partial D(\dot{p},-\dot{\alpha}_1,-\dot{\alpha}_2,\dot{\kappa}),\\ \sqcup_C(\sigma,\chi_1,\chi_2,X) + D(\dot{p},-\dot{\alpha}_1,-\dot{\alpha}_2,\dot{\kappa}) = ((\sigma,\dot{p})) - ((\chi_1,\dot{\alpha}_1)) - ((\chi_2,\dot{\alpha}_2)) + ((X,\dot{\kappa})). \end{cases}$$

Let us express the flow rule $(14)_1$ in terms of the plastic multiplier λ. Assuming that G is continuous in C and noting that $\sqcup_C(\boldsymbol{\sigma},X) = \sqcup_{\Re^-}[G(\boldsymbol{\sigma},X)]$ where $\Re^-$ collects the nonpositive scalars, a subdifferential rule contributed in Romano (1995) yields

$$\partial \sqcup_C (\boldsymbol{\sigma},X) = \partial(\sqcup_{\Re^-} \circ G)(\boldsymbol{\sigma},X) = \partial \sqcup_{\Re^-} [G(\boldsymbol{\sigma},X)]\, \mathrm{d}G(\boldsymbol{\sigma},X) \quad \forall(\boldsymbol{\sigma},X) \in C.$$

Since $\partial \sqcup_{\Re^-} [G(\boldsymbol{\sigma},X)] = N_{\Re^-}[G(\boldsymbol{\sigma},X)]$, we have that the flow rule $(14)_1$ can be rewritten in the following equivalent forms:

$$\begin{aligned} &(\dot{\boldsymbol{p}},\dot{\kappa}) \in N_C(\boldsymbol{\sigma},X) = \partial \sqcup_C (\boldsymbol{\sigma},X),\\ &(\dot{\boldsymbol{p}},\dot{\kappa}) = \lambda\, \mathrm{d}G(\boldsymbol{\sigma},X) \quad \text{with } \lambda \in N_{\Re^-}[G(\boldsymbol{\sigma},X)] = \partial \sqcup_{\Re^-} [G(\boldsymbol{\sigma},X)],\\ &(\dot{\boldsymbol{p}},\dot{\kappa}) = \lambda\, \mathrm{d}G(\boldsymbol{\sigma},X) \quad \text{with } \lambda \geqslant 0,\ \ G(\boldsymbol{\sigma},X) \leqslant 0,\ \ \lambda G(\boldsymbol{\sigma},X) = 0, \end{aligned} \tag{16}$$

where λ is the plastic multiplier.

Recalling the expression (12) of the yield mode, the relation $(16)_3$ is explicitly given by

$$\begin{aligned} \dot{p} &= \lambda \, \mathrm{d}_\sigma G(\sigma, \chi_1, \chi_2, X) = \lambda \, \mathrm{d}_\sigma g(\sigma, \chi_1), \\ -\dot{\alpha}_1 &= \lambda \, \mathrm{d}_{\chi_1} G(\sigma, \chi_1, \chi_2, X) = \lambda \, \mathrm{d}_{\chi_1} g(\sigma, \chi_1), \\ -\dot{\alpha}_2 &= \lambda \, \mathrm{d}_{\chi_2} G(\sigma, \chi_1, \chi_2, X) = -\lambda, \\ \dot{\kappa} &= \lambda \, \mathrm{d}_X G(\sigma, \chi_1, \chi_2, X) = -\lambda \end{aligned} \tag{17}$$

under the complementarity conditions

$$\lambda \geqslant 0, \qquad g(\sigma, \chi_1) - \chi_2 - X - \sigma_0 \leqslant 0, \qquad \lambda[g(\sigma, \chi_1) - \chi_2 - X - \sigma_0] = 0. \tag{18}$$

As a result, the rate of the kinematic internal variable $\dot{\alpha}_2$ coincides with the plastic multiplier so that the parametric representation of $\dot{\xi}$ is given by $\dot{\xi} = \mathbf{R}\lambda$. Since in the elastic range $\kappa = \alpha_2 = 0$, the relations $(17)_3$ and $(17)_4$ yield the equality $\kappa = -\alpha_2$.

3.1. Nonlocal form of dissipation and of Lagrangian functional

It is well known that the *principle of the maximum dissipation* of Hill (Hill, 1950; Simo, 1988) plays a central role in local standard plasticity. This subsection shows that an analogous principle can be assessed for the proposed model of nonlocal standard plasticity.

Proposition 1 (Maximum dissipation). *Let* $(\boldsymbol{\sigma}, X)$ *and* $(\dot{\boldsymbol{p}}, \dot{\kappa})$ *fulfill the flow rule* (13). *The dissipation* $D(\dot{\boldsymbol{p}}, \dot{\kappa})$ *attains its maximum at the point* $(\boldsymbol{\sigma}, X)$ *and can be written in the form*

$$D(\dot{\boldsymbol{p}}, \dot{\kappa}) = ((\boldsymbol{\sigma}, \dot{\boldsymbol{p}})) + ((X, \dot{\kappa})) = ((\sigma, \dot{p})) - ((\chi_1, \dot{\alpha}_1)) - ((\chi_2, \dot{\alpha}_2)) + ((X, \dot{\kappa})). \tag{19}$$

Proof. The pairs $(\boldsymbol{\sigma}, X)$ and $(\dot{\boldsymbol{p}}, \dot{\kappa})$ fulfill the flow rule (13) so that the Fenchel's equality $(14)_3$ is met. Hence we have

$$D(\dot{\boldsymbol{p}}, \dot{\kappa}) = ((\boldsymbol{\sigma}, \dot{\boldsymbol{p}})) + ((X, \dot{\kappa})) - \sqcup_C(\boldsymbol{\sigma}, X) = ((\boldsymbol{\sigma}, \dot{\boldsymbol{p}})) + ((X, \dot{\kappa}))$$

since the indicator $\sqcup_C$ is zero being the static variables admissible, i.e. $(\boldsymbol{\sigma}, X) \in C$. □

Following the arguments presented in Romano et al. (1992) for local plasticity, it can be proved that the dissipation D is nonnegative if the origin of the strain space $\widehat{D} \times \mathscr{Y}$ belongs to the elastic domain C.

The nonlocal dissipation for the Cauchy model is then given by

$$D(\dot{p}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\kappa}) = \int_\Omega \sigma \cdot \dot{p} \, \mathrm{d}\mathbf{x} - \int_\Omega \chi_1 \cdot \dot{\alpha}_1 \, \mathrm{d}\mathbf{x} - \int_\Omega \frac{\chi(\mathbf{x})}{V(\mathbf{x})} \cdot \left[\int_\Omega \beta_{\mathbf{x}}(\mathbf{y}) \dot{\alpha}_2(\mathbf{y}) \right] \mathrm{d}\mathbf{x} + \int_\Omega X \dot{\kappa} \, \mathrm{d}\mathbf{x}$$

and, for gradient plasticity, it turns out to be

$$D(\dot{p}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\kappa}) = \int_\Omega \sigma \cdot \dot{p} \, \mathrm{d}\mathbf{x} - \int_\Omega \chi_1 \cdot \dot{\alpha}_1 \, \mathrm{d}\mathbf{x} - \int_\Omega c\chi \cdot \nabla \dot{\alpha}_2 \, \mathrm{d}\mathbf{x} + \int_\Omega X \dot{\kappa} \, \mathrm{d}\mathbf{x}.$$

Let us now derive the expression of the *Lagrangian* L associated with the dissipation D. Recalling that $\sqcup_C(\boldsymbol{\sigma}, X) = \sqcup_{\Re^-}[G(\boldsymbol{\sigma}, X)]$ and noting that $\sqcup_{\Re^-}[G(\boldsymbol{\sigma}, X)]$ and $\sqcup_{\Re^+}(\lambda)$ are conjugate functionals (Hiriart-Urruty and Lemarechal, 1993), it turns out to be

$$\sqcup_C(\boldsymbol{\sigma}, X) = \sqcup_{\Re^-}[G(\boldsymbol{\sigma}, X)] = \sup_{\bar{\lambda}} \{ \bar{\lambda} G(\boldsymbol{\sigma}, X) - \sqcup_{\Re^+}(\bar{\lambda}) \} = -\inf_{\bar{\lambda}} \{ -\bar{\lambda} G(\boldsymbol{\sigma}, X) + \sqcup_{\Re^+}(\bar{\lambda}) \}, \tag{20}$$

where the equality $\sup(\bullet) = -\inf(-/\bullet)$ has been used.

Accordingly the expression (15) of the dissipation D associated with a plastic flow $(\dot{\boldsymbol{p}}, \dot{\kappa})$ becomes

$$D(\dot{\boldsymbol{p}}, \dot{\kappa}) = \sup_{\bar{\sigma},\bar{X}}\{((\bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\bar{X}, \dot{\kappa})) - \sqcup_C(\bar{\boldsymbol{\sigma}}, \bar{X})\} = \sup_{\bar{\sigma},\bar{X}} \inf_{\bar{\lambda}}\{((\bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\bar{X}, \dot{\kappa})) - \bar{\lambda}G(\bar{\boldsymbol{\sigma}}, \bar{X}) + \sqcup_{\Re+}(\bar{\lambda})\}$$
$$= \sup_{\bar{\sigma},\bar{X}} \inf_{\bar{\lambda}} L(\bar{\lambda}, \bar{\boldsymbol{\sigma}}, \bar{X}),$$

where $L(\bar{\lambda}, \bar{\boldsymbol{\sigma}}, \bar{X}) = ((\bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\bar{X}, \dot{\kappa})) - \bar{\lambda}G(\bar{\boldsymbol{\sigma}}, \bar{X}) + \sqcup_{\Re+}(\bar{\lambda})$ denotes the Lagrangian, associated with D, which turns out to be concave in $(\bar{\boldsymbol{\sigma}}, \bar{X})$ and convex in $\bar{\lambda}$ for any given plastic flow $(\dot{\boldsymbol{p}}, \dot{\kappa})$. A saddle point is attained for a pair $(\lambda, \boldsymbol{\sigma}, X)$ such that the plastic flow $(\dot{\boldsymbol{p}}, \dot{\kappa})$ fulfils the normality rule (16) at $(\boldsymbol{\sigma}, X)$:

$$D(\dot{\boldsymbol{p}}, \dot{\kappa}) = \sup_{\bar{\sigma},\bar{X}} \inf_{\bar{\lambda} \geqslant 0}\{((\bar{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\bar{X}, \dot{\kappa})) - \bar{\lambda}G(\bar{\boldsymbol{\sigma}}, \bar{X})\} = ((\boldsymbol{\sigma}, \dot{\boldsymbol{p}})) + ((X, \dot{\kappa})).$$

4. The constitutive rate model

The constitutive model of nonlocal and gradient plasticity can be formulated by considering the additivity of strains, the constitutive relations (7) and the flow rule (14):

$$\begin{cases} \boldsymbol{\varepsilon} = \boldsymbol{e} + \boldsymbol{p} & \text{(additivity of strains)}, \\ (\dot{\boldsymbol{p}}, \dot{\kappa}) \in N_C(\boldsymbol{\sigma}, X) & \text{(flow rule)}, \\ (\boldsymbol{\sigma}, -X) = \mathrm{d}\Phi(\boldsymbol{e}, \kappa) & \text{(elastic relation)}. \end{cases} \tag{21}$$

To define the relevant rate model, it is compelling to reformulate the constitutive relations (21) in terms of the rate of the state fields.

The relation between the stress rate $(\dot{\boldsymbol{\sigma}}, \dot{X})$ and the plastic flow $(\dot{\boldsymbol{p}}, \dot{\kappa})$ can be achieved by considering the Prager's consistency condition:

$$((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\dot{X}, \dot{\kappa})) = 0 \quad \forall (\dot{\boldsymbol{p}}, \dot{\kappa}) \in N_C(\boldsymbol{\sigma}, X), (\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}}) \in T_C(\boldsymbol{\sigma}, X), \tag{22}$$

where $T_C(\boldsymbol{\sigma}, X)$ is the tangent cone to the elastic domain C at the point $(\boldsymbol{\sigma}, X)$. If $(\boldsymbol{\sigma}, X)$ belong to the interior of C, the generalized stress rate $(\dot{\boldsymbol{\sigma}}, \dot{X})$ is arbitrary. On the contrary if $(\boldsymbol{\sigma}, X)$ belongs to the boundary of C, the tangent cone turns out to be a proper subset of $\widehat{S} \times \mathscr{Y}'$.

The normality rule $(21)_2$ and the Prager's consistency condition (22) can be collected in a unique relation. In fact, setting $\mathscr{T} = T_C(\boldsymbol{\sigma}, X)$ and $\mathscr{N} = N_C(\boldsymbol{\sigma}, X)$, condition (22) is equivalent to the equality:

$$\sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) + \sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\dot{X}, \dot{\kappa})). \tag{23}$$

By virtue of the equality $\sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}) = \sqcup^*_{\mathscr{T}}(\dot{\boldsymbol{p}}, \dot{\kappa})$ proved in Appendix A, the relation (23) can be rewritten in the three equivalent forms:

$$\begin{aligned} &\sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) + \sqcup^*_{\mathscr{T}}(\dot{\boldsymbol{p}}, \dot{\kappa}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\dot{X}, \dot{\kappa})), \\ &(\dot{\boldsymbol{\sigma}}, \dot{X}) \in \partial \sqcup^*_{\mathscr{T}}(\dot{\boldsymbol{p}}, \dot{\kappa}) = \partial \sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}), \\ &(\dot{\boldsymbol{p}}, \dot{\kappa}) \in \partial \sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) = N_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}). \end{aligned} \tag{24}$$

The rate elastic constitutive relation can be obtained from the elastic relation $(21)_3$:

$$(\dot{\boldsymbol{\sigma}}, -\dot{X}) = \mathrm{d}^2\Phi(\boldsymbol{e}, \kappa)(\dot{\boldsymbol{e}}, \dot{\kappa}) \iff \begin{cases} \dot{\sigma} = \mathrm{d}^2\Phi_{\mathrm{el}}(e)\dot{e} = \mathbf{E}\dot{e}, \\ \dot{\chi}_1 = \mathrm{d}^2_{\alpha_1}\Phi_L(\alpha_1, \kappa)\dot{\alpha}_1 = \mathbf{H}_1\dot{\alpha}_1, \\ \dot{\chi}_2 = \mathrm{d}^2_{\alpha_2}\Phi_{NL}(\xi(\alpha_2))\dot{\alpha}_2 = [\mathbf{R}'\,\mathrm{d}^2\Phi_{NL}(\xi)\mathbf{R}]\dot{\alpha}_2 = \mathsf{H}_2\dot{\alpha}_2, \\ -\dot{X} = \mathrm{d}^2_{\kappa}\Phi_L(\alpha_1, \kappa)\dot{\kappa} = \mathbf{H}\dot{\kappa}, \end{cases} \tag{25}$$

where $\mathbf{E} = d^2\Phi_{el}(e)$ denotes the tangent elastic modulus, $\mathbf{H}_1 = d^2_{\alpha_1}\Phi_L(\alpha_1, \kappa)$ is the tangent kinematic hardening modulus, $\mathsf{H}_2 = \mathbf{R}' d^2\Phi_{NL}(\xi)\mathbf{R}$ provides the tangent isotropic hardening modulus and $\mathbf{H} = d^2_{\kappa}\Phi_L(\alpha_1, \kappa)$ represents the tangent isotropic softening modulus. For a linear behaviour of the type (8), we have $\mathsf{H}_2 = \mathbf{R}' h \mathbf{R}$.

Denoting by $\mathscr{H} = \mathrm{diag}[\mathbf{E}, \mathbf{H}_1, \mathsf{H}_2, \mathbf{H}]$ the matrix collecting the elastic and hardening/softening tangent moduli, the relations (25) can be expressed in the form $(\dot{\boldsymbol{\sigma}}, -\dot{X}) = d\Psi(\dot{\boldsymbol{e}}, \dot{\kappa})$ by introducing the saddle (convex in $\dot{\boldsymbol{e}}$ and concave in $\dot{\kappa}$) *rate elastic potential* $\Psi : \widehat{\mathscr{D}} \times \mathscr{Y} \to \bar{\Re}$ (where $\overline{\Re} = \{-\infty \cup \Re \cup +\infty\}$):

$$\begin{aligned}\Psi(\dot{\boldsymbol{e}}, \dot{\kappa}) &= \tfrac{1}{2} \prec \mathscr{H}(\dot{\boldsymbol{e}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\kappa}), (\dot{\boldsymbol{e}}, \dot{\alpha}_1, \dot{\alpha}_2, \dot{\kappa}) \succ \\ &= \tfrac{1}{2}((\mathbf{E}\dot{e}, \dot{e})) + \tfrac{1}{2}((\mathbf{H}_1\dot{\alpha}_1, \dot{\alpha}_1)) + \tfrac{1}{2}((\mathsf{H}_2\dot{\alpha}_2, \dot{\alpha}_2)) + \tfrac{1}{2}((\mathbf{H}\dot{\kappa}, \dot{\kappa})).\end{aligned} \tag{26}$$

The constitutive model of nonlocal and gradient rate plasticity is then given by

$$\begin{cases} \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{e}} + \dot{\boldsymbol{p}} & \text{(additivity of strain rates)}, \\ (\dot{\boldsymbol{p}}, \dot{\kappa}) \in N_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) & \text{(rate flow rule)}, \\ (\dot{\boldsymbol{\sigma}}, -\dot{X}) = d\Psi(\dot{\boldsymbol{e}}, \dot{\kappa}) & \text{(rate elastic relation)}. \end{cases} \tag{27}$$

In order to derive a variational formulation pertaining to the proposed nonlocal and gradient elastoplastic model, it is compelling to consider alternative expressions of the rate elastic relation. To this end we introduce the conjugate saddle functional $\Psi^* : \widehat{\mathscr{S}} \times \mathscr{Y}' \to \bar{\Re}$ which represents the *complementary* rate elastic potential defined by

$$\Psi^*(\dot{\boldsymbol{\sigma}}, \dot{X}) = \inf_{\dot{\kappa}} \sup_{\dot{\boldsymbol{e}}} \{((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})) + ((\dot{X}, \dot{\kappa})) - \Psi(\dot{\boldsymbol{e}}, \dot{\kappa}), \}$$

and the convex functionals $\Xi : \widehat{\mathscr{S}} \times \mathscr{Y} \to \Re \cup +\infty\}$ and $\Xi^* : \widehat{\mathscr{D}} \times \mathscr{Y}' \to \Re \cup +\infty\}$, associated with Ψ and Ψ^*, defined by

$$\begin{aligned} \Xi(\dot{\boldsymbol{\sigma}}, \dot{\kappa}) &= -\inf_{\dot{X}} \{((\dot{X}, \dot{\kappa})) - \Psi^*(\dot{\boldsymbol{\sigma}}, \dot{X})\} = \sup_{\dot{\boldsymbol{e}}} \{((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})) - \Psi(\dot{\boldsymbol{e}}, \dot{\kappa})\}, \\ \Xi^*(\dot{\boldsymbol{e}}, -\dot{X}) &= -\inf_{\dot{\kappa}} \{((\dot{X}, \dot{\kappa})) - \Psi(\dot{\boldsymbol{e}}, \dot{\kappa})\} = \sup_{\dot{\boldsymbol{\sigma}}} \{((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})) - \Psi^*(\dot{\boldsymbol{\sigma}}, \dot{X})\}. \end{aligned} \tag{28}$$

The rate elastic relation $(27)_3$ can then be inverted according to the following equivalent expressions:

$$\begin{aligned} &(\dot{\boldsymbol{\sigma}}, -\dot{X}) = d\Psi(\dot{\boldsymbol{e}}, \dot{\kappa}), \qquad (\dot{\boldsymbol{e}}, \dot{\kappa}) = d\Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}), \\ &(\dot{\boldsymbol{e}}, \dot{X}) = d\Xi(\dot{\boldsymbol{\sigma}}, \dot{\kappa}), \qquad (\dot{\boldsymbol{\sigma}}, \dot{\kappa}) = d\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) \end{aligned} \tag{29}$$

which can be equivalently rewritten in terms of Fenchel's equalities:

$$\begin{aligned} &-\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) + \Psi(\dot{\boldsymbol{e}}, \dot{\kappa}) = -((\dot{X}, \dot{\kappa})), \qquad \Xi(\dot{\boldsymbol{\sigma}}, \dot{\kappa}) + \Psi(\dot{\boldsymbol{e}}, \dot{\kappa}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})), \\ &-\Xi(\dot{\boldsymbol{\sigma}}, \dot{\kappa}) + \Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}) = -((\dot{X}, \dot{\kappa})), \qquad \Xi^*(\dot{\boldsymbol{e}}, \dot{X}) + \Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})), \\ &\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) + \Xi(\dot{\boldsymbol{\sigma}}, \dot{\kappa}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})) + ((\dot{X}, \dot{\kappa})) \qquad \Psi(\dot{\boldsymbol{e}}, \dot{\kappa}) + \Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}) = ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}})) - ((\dot{X}, \dot{\kappa})). \end{aligned} \tag{30}$$

4.1. Variational formulations for a given total strain rate

The rate constitutive model (27) is governed by a monotone multivalued operator (see Appendix B) since the equality $N_{\mathscr{T}} = \partial \sqcup_{\mathscr{T}}$ holds. Accordingly a direct formulation of the constitutive variational principles can be obtained by resorting to the potential theory for monotone multivalued operators contributed in Romano et al. (1993).

The variational formulation in terms of the state variables $(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\kappa})$, for a given total strain rate $\dot{\boldsymbol{\varepsilon}}$, is provided in the next statement. The related proof is reported in Appendix B.

Proposition 2. *For a given* $\dot{\boldsymbol{\varepsilon}}$, *the quintet* $(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa})$ *is a solution of the convex optimization problem*:

$$\min_{\dot{\boldsymbol{e}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa}} \operatorname{stat}_{\dot{\boldsymbol{\sigma}}} \boldsymbol{\Sigma}(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa}),$$

where

$$\boldsymbol{\Sigma}(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa}) = \Xi^*(\dot{\boldsymbol{e}}, \dot{X}) + \sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}) - ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}} + \dot{\boldsymbol{p}})) - ((\dot{X}, \dot{\kappa})) + ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}})) \tag{31}$$

if and only if it is a solution of the nonlocal elastoplastic rate model (27).

The expression (31) shows that the potential $\boldsymbol{\Sigma}$ turns out to be convex in the state variables $(\dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa})$ and linear in $\dot{\boldsymbol{\sigma}}$. Accordingly a stationarity point for $\boldsymbol{\Sigma}$ can be found by performing the minimum of $\boldsymbol{\Sigma}$ with respect to the variables $(\dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa})$ and by enforcing the stationarity condition with respect to $\dot{\boldsymbol{\sigma}}$.

It is proved in Appendix B that a direct integration of the constitutive relations (27) provides the potential $\boldsymbol{\Sigma}$. Let us now show that the state variables $(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa})$ provide a stationarity point for $\boldsymbol{\Sigma}$ if and only if the constitutive relations (27) are recovered. In fact we have

$$(\boldsymbol{o}, \boldsymbol{o}, 0, \boldsymbol{o}, 0) \in \partial\boldsymbol{\Sigma}(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa}) \iff \begin{cases} (\boldsymbol{o}, 0) \in \partial_{(\dot{\boldsymbol{e}},\dot{X})} \boldsymbol{\Sigma} \Leftrightarrow \begin{bmatrix} \boldsymbol{o} \\ 0 \end{bmatrix} = \mathrm{d}\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) - \begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ \dot{\kappa} \end{bmatrix}, \\ (\boldsymbol{o}, 0) \in \partial_{(\dot{\boldsymbol{p}},\dot{\kappa})} \boldsymbol{\Sigma} \Leftrightarrow \begin{bmatrix} \boldsymbol{o} \\ 0 \end{bmatrix} \in \partial \sqcup_{\mathscr{N}} (\dot{\boldsymbol{p}}, \dot{\kappa}) - \begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ \dot{X} \end{bmatrix}, \\ \boldsymbol{o} \in \partial_{\dot{\boldsymbol{\sigma}}} \boldsymbol{\Sigma} \Leftrightarrow \boldsymbol{o} = -(\dot{\boldsymbol{e}} + \dot{\boldsymbol{p}}) + \dot{\boldsymbol{\varepsilon}}, \end{cases} \tag{32}$$

where $\partial_{(\bullet)}\boldsymbol{\Sigma}$ denotes the subdifferential of $\boldsymbol{\Sigma}$ with respect to $\bullet$. If $\boldsymbol{\Sigma}$ is differentiable with respect to the variable $\bullet$, the subdifferential turns out to be the usual differential. From $(32)_1$ and recalling the equivalent expressions (29), we have

$$(\dot{\boldsymbol{\sigma}}, \dot{\kappa}) = \mathrm{d}\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) \iff (\dot{\boldsymbol{\sigma}}, -\dot{X}) = \mathrm{d}\Psi(\dot{\boldsymbol{e}}, \dot{\kappa})$$

so that the rate elastic relation $(27)_3$ has been obtained. From $(32)_2$ and recalling the relations (24), we have

$$(\dot{\boldsymbol{\sigma}}, \dot{X}) \in \partial \sqcup_{\mathscr{N}} (\dot{\boldsymbol{p}}, \dot{\kappa}) \iff (\dot{\boldsymbol{p}}, \dot{\kappa}) \in \partial \sqcup_{\mathscr{T}} (\dot{\boldsymbol{\sigma}}, \dot{X}) = N_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X})$$

which coincides to the rate flow rule $(27)_2$. Finally the additivity of strain rates is given by $(32)_3$ and the constitutive model is recovered. An analogous procedure can be followed to prove the variational formulations hereafter obtained.

Several variational formulations can be obtained by enforcing the fulfilment of the constitutive relations (27) in the expression of the potential $\boldsymbol{\Sigma}$. If the additive decomposition $(27)_1$ is fulfilled and the rate elastic relation is met in the equivalent form $(30)_1$, the potential (31) becomes

$$\boldsymbol{\Sigma}_1(\dot{\boldsymbol{p}}, \dot{\kappa}) = \Psi(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{p}}, \dot{\kappa}) + \sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}) \tag{33}$$

which is convex in $\dot{\boldsymbol{p}}$ and locally subdifferentiable (Romano, 1995) in $\dot{\kappa}$.

Let the rate flow rule $(27)_2$ and the rate elastic relation $(27)_3$ be fulfilled in the form of Fenchel's equalities (23) and $(30)_4$, the potential (31) thus becomes

$$\boldsymbol{\Sigma}_2(\dot{\boldsymbol{\sigma}}, \dot{X}) = -\Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}) - \sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) + ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}})) \tag{34}$$

which is concave in $\dot{\boldsymbol{\sigma}}$ and locally subdifferentiable in $\dot{X}$.

Hence the next statement can be inferred.

Proposition 3. *For a given* $\dot{\boldsymbol{\varepsilon}}$, *we have*

(i) *the pair* $\{\dot{\boldsymbol{p}}, \dot{\kappa}\}$ *is a solution of the optimization problem*: $\min_{\dot{\boldsymbol{p}}} \operatorname{stat}_{\dot{\kappa}} \boldsymbol{\Sigma}_1(\dot{\boldsymbol{p}}, \dot{\kappa})$,

(ii) *the pair* $\{\dot{\boldsymbol{\sigma}}, \dot{X}\}$ *is a solution of the optimization problem*: $\max_{\dot{\boldsymbol{\sigma}}} \operatorname{stat}_{\dot{X}} \boldsymbol{\Sigma}_2(\dot{\boldsymbol{\sigma}}, \dot{X})$, *if and only if it is a solution of the nonlocal and gradient elastoplastic rate model* (27).

Additional variational formulations depending on different combinations of the state variables can be obtained following the same reasoning.

It is useful to derive a constitutive variational formulation in terms of the plastic multiplier λ in order to perform, in Section 5.2, a comparison with an analogous functional provided in Borino et al. (1999) in the case of linear elastic and hardening behaviour.

To this end, we note that the indicator $\sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}},\dot{\kappa})$, appearing in (33), requires that the plastic flow fulfils the flow rule, that is $(\dot{\boldsymbol{p}},\dot{\kappa}) \in N_C(\boldsymbol{\sigma},X)$. Recalling the expression (26) of the rate elastic energy Ψ and the expression (17) of the flow rule, the functional $\boldsymbol{\Sigma}_1(\dot{\boldsymbol{p}},\dot{\kappa})$ can be rewritten in terms of the plastic multiplier λ in the form:

$$\begin{aligned}\boldsymbol{\Sigma}_3(\lambda) = {} & \tfrac{1}{2}((\mathbf{E}\dot{\varepsilon},\dot{\varepsilon})) - ((\mathbf{E}\dot{\varepsilon},\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1))) + \tfrac{1}{2}((\mathbf{E}\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1),\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1))) \\ & + \tfrac{1}{2}((\mathbf{H}_1\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1),\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1))) + \tfrac{1}{2}((\mathsf{H}_2\lambda,\lambda)) + \tfrac{1}{2}((\mathbf{H}\lambda,\lambda))\end{aligned} \tag{35}$$

subject to the conditions $\lambda \geqslant 0, G(\boldsymbol{\sigma}(\lambda),X(\lambda)) \leqslant 0, \lambda G(\boldsymbol{\sigma}(\lambda),X(\lambda)) = 0$.

By dropping the constant term $((\mathbf{E}\dot{\varepsilon},\dot{\varepsilon}))$, we have

Proposition 4. *For a given* $\dot{\varepsilon}$, *the plastic multiplier* λ *is a solution of the optimization problem*:

$$\mathrm{stat}\{\Pi(\lambda) \text{ s.t. } \lambda \geqslant 0,\ G(\boldsymbol{\sigma}(\lambda),X(\lambda)) \leqslant 0,\ \lambda G(\boldsymbol{\sigma}(\lambda),\ X(\lambda)) = 0\},$$

where

$$\begin{aligned}\Pi(\lambda) = {} & \tfrac{1}{2}((\mathbf{E}\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1),\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1))) - ((\mathbf{E}\dot{\varepsilon},\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1))) + \tfrac{1}{2}((\mathbf{H}_1\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1),\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1))) \\ & + \tfrac{1}{2}((\mathsf{H}_2\lambda,\lambda)) + \tfrac{1}{2}((\mathbf{H}\lambda,\lambda))\end{aligned}$$

if and only if it is a solution of the nonlocal elastoplastic rate model (27).

Here "s.t." stands for "subject to". The specialization of the functional Π to the case of nonlocal plasticity for the Cauchy model is

$$\begin{aligned}\Pi(\lambda) = {} & \frac{1}{2}\int_\Omega \mathbf{E}\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\cdot\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{E}\dot{\varepsilon}\cdot\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mathbf{H}_1\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1)\cdot\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} \\ & + \frac{1}{2}\int_\Omega \frac{h}{V(\mathbf{x})^2}\left[\int_\Omega \beta_{\mathbf{x}}(\mathbf{y})\lambda(\mathbf{y})\,\mathrm{d}\mathbf{y}\right]^2 \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mathbf{H}\lambda^2\,\mathrm{d}\mathbf{x},\end{aligned}$$

and to the case of gradient plasticity is

$$\begin{aligned}\Pi(\lambda) = {} & \frac{1}{2}\int_\Omega \mathbf{E}\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\cdot\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{E}\dot{\varepsilon}\cdot\lambda\,\mathrm{d}_\sigma g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mathbf{H}_1\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1)\cdot\lambda\,\mathrm{d}_{\chi_1} g(\sigma,\chi_1)\,\mathrm{d}\mathbf{x} \\ & + \frac{1}{2}\int_\Omega hc^2\nabla\lambda\cdot\nabla\lambda\,\mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mathbf{H}\lambda^2\,\mathrm{d}\mathbf{x}.\end{aligned}$$

5. The structural rate problem for nonlocal plasticity

Let us now analyse the rate response of an elastoplastic structural problem having a nonlocal constitutive behaviour in the form previously defined. Displacements are assumed to belong to the Sobolev space $\mathscr{U} = H^m(\Omega)$ of fields which are square integrable in Ω together with their distributional derivatives up to the order m (Brezis, 1983). Conforming displacement fields fulfil linear constraint conditions and belong to a closed linear subspace $\mathscr{L} \subset \mathscr{U}$.

The kinematic operator $\mathbf{B} \in \text{Lin}\{\mathscr{U}, \mathscr{D}\}$ is a bounded linear operator from $\mathscr{U}$ to the Hilbert space of square integrable strain fields. Denoting by $\mathscr{F}$ the subspace of external forces, which is dual of $\mathscr{U}$, the equilibrium operator $\mathbf{B}' \in \text{Lin}\{\mathscr{S}, \mathscr{F}\}$ is dual of $\mathbf{B}$. The symbol $\langle \bullet, \bullet \rangle$ denotes the duality pairing between $\mathscr{U}$ and its dual $\mathscr{F}$. Let $\ell = \{\mathbf{t}, \mathbf{b}\} \in \mathscr{F}$ be the external loads where $\mathbf{t}$ and $\mathbf{b}$ denote tractions and body forces. No imposed strains and displacements are considered for simplicity but they may be easily included.

The equilibrium equation, between external forces f and stresses σ, and the compatibility condition, between strains ε and displacements u, are given by

$$f = \mathbf{B}'\sigma \quad \text{where } f \in \mathscr{F},\ \sigma \in \mathscr{S}, \quad \varepsilon \in \mathbf{B}u \quad \text{where } \varepsilon \in \mathscr{D}, u \in \mathscr{U}.$$

The relation between reactions and displacements is assumed to be given by

$$r \in \partial \Upsilon(u) \quad \text{or equivalently } u \in \partial \Upsilon^*(r),$$

where $\Upsilon : \mathscr{U} \to \Re \cup \{-\infty\}$ is a concave functional and $\Upsilon^* : \mathscr{F} \to \Re \cup \{-\infty\}$ represents its conjugate (Hiriart-Urruty and Lemarechal, 1993). Different expressions can be given to the functional Υ depending on the type of external constraints such as bilateral, unilateral, elastic or convex. A survey of the particular expression assumed by the functional Υ in each of these cases can be found in Romano (2002). For future reference we report the expressions of Υ and Υ^* in the case of external frictionless bilateral constraints with homogeneous boundary conditions. Denoting by $\mathscr{L}$ the subspace of conforming displacements and by $R = \mathscr{L}^{\perp}$ the subspace of the external constraint reactions, it turns out to be

$$\Upsilon(u) = \sqcap_{\mathscr{L}}(u) = \begin{cases} 0 & \text{if } u \in \mathscr{L}, \\ -\infty & \text{otherwise,} \end{cases} \quad \Upsilon^*(r) = \sqcap_{\mathscr{L}^{\perp}}(r) = \begin{cases} 0 & \text{if } r \in \mathscr{L}^{\perp}, \\ -\infty & \text{otherwise,} \end{cases}$$

being $\perp$ the orthogonal complement. Accordingly the relation $r \in \partial \Upsilon(u)$ is equivalent to state $u \in \mathscr{L}$ and $r \in R = \mathscr{L}^{\perp}$, i.e. $\langle r, v \rangle = 0$ for any conforming displacement $v \in \mathscr{L}$.

Defining the pair of dual operators $\overline{\mathbf{B}} = [\mathbf{B} \quad 0 \quad 0]^T : U \to \widehat{\mathscr{D}}$ and $\overline{\mathbf{B}}' = [\mathbf{B}' \quad 0 \quad 0] : \widehat{\mathscr{S}} \to \mathscr{F}$, the relations governing the nonlocal and gradient structural rate problem of the body Ω for a given load rate $\dot{\ell}$, starting from a known state, are given by

$$\begin{cases} \overline{\mathbf{B}}'\dot{\boldsymbol{\sigma}} = \dot{\ell} + \dot{r} & \text{(equilibrium rate)}, \\ \overline{\mathbf{B}}\dot{u} = \dot{\boldsymbol{e}} + \dot{\boldsymbol{p}} & \text{(compatibility rate)}, \\ (\dot{\boldsymbol{\sigma}}, \dot{\kappa}) = \mathrm{d}\Xi^*(\dot{\boldsymbol{e}}, \dot{X}) & \text{(rate elastic relation)}, \\ (\dot{\boldsymbol{\sigma}}, \dot{X}) \in \partial \sqcup_{\mathscr{N}} (\dot{\boldsymbol{p}}, \dot{\kappa}) & \text{(rate flow rule)}, \\ \dot{u} \in \partial \Upsilon^*(\dot{r}) & \text{(rate external relation)}. \end{cases} \tag{36}$$

The structural problem can be recast in the following operator form:

$$\begin{bmatrix} 0 \\ \circ \\ \circ \\ \\ 0 \\ \circ \\ \\ 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} 0 & \overline{\mathbf{B}}' & 0 & 0 & 0 & 0 & -I_{\mathscr{F}} \\ \overline{\mathbf{B}} & 0 & -I_{\widehat{\mathscr{D}}} & 0 & -I_{\widehat{\mathscr{D}}} & 0 & 0 \\ 0 & -I_{\widehat{\mathscr{S}}} & & & 0 & 0 & 0 \\ & & & \mathrm{d}\Xi^* & & & \\ 0 & 0 & & & 0 & -I_{\mathscr{Y}} & 0 \\ 0 & -I_{\widehat{\mathscr{S}}} & 0 & 0 & & & 0 \\ & & & & \partial \sqcup_{\mathscr{N}} & & \\ 0 & 0 & 0 & -I_{\mathscr{Y}'} & & & 0 \\ -I_{\mathscr{U}} & 0 & 0 & 0 & 0 & 0 & \partial \Upsilon^* \end{bmatrix} \begin{bmatrix} \dot{u} \\ \dot{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{e}} \\ \\ \dot{X} \\ \dot{\boldsymbol{p}} \\ \\ \dot{\kappa} \\ \dot{r} \end{bmatrix} - \begin{bmatrix} \dot{\ell} \\ \circ \\ \circ \\ \\ 0 \\ \circ \\ \\ 0 \\ 0 \end{bmatrix},$$

where $I_\bullet$ denotes the identity in the space $\bullet$.

The conservativity of the above structural operator is based on the conservativity of the constitutive operator $\mathbf{C}$ and can be proved following a reasoning similar to the one reported in Appendix B. A direct integration of the structural operator provides the following potential in the complete set of state variables is thus obtained:

$$\mathbf{\Omega}(\dot{u},\dot{\boldsymbol{\sigma}},\dot{\boldsymbol{e}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa},\dot{r}) = \Xi^*(\dot{\boldsymbol{e}},\dot{X}) + \sqcup_{\mathcal{N}}(\dot{\boldsymbol{p}},\dot{\kappa}) + \Upsilon^*(r) + ((\dot{\boldsymbol{\sigma}},\overline{\mathbf{B}}\dot{u})) - ((\dot{\boldsymbol{\sigma}},\dot{\boldsymbol{e}}+\dot{\boldsymbol{p}})) - ((\dot{X},\dot{\kappa})) + \langle\dot{\ell}+\dot{r},\dot{u}\rangle. \tag{37}$$

The potential $\mathbf{\Omega}$ is linear in $(\dot{u},\dot{\boldsymbol{\sigma}})$, convex in $(\dot{\boldsymbol{e}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa})$ and concave in $\dot{r}$. We then have

Proposition 5. *A set* $(\dot{u},\dot{\boldsymbol{\sigma}},\dot{\boldsymbol{e}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa},\dot{r})$ *is a solution of the nonlocal and gradient rate elastoplastic structural problem if and only if it turns out to be a stationarity point for* $\mathbf{\Omega}$.

A family of potentials can be recovered from $\mathbf{\Omega}$ by enforcing the relations (36). The stationary points of these potentials provide a solution of the structural problem.

5.1. Variational principles

Let us now derive a variational principle for nonlocal and gradient plasticity which specializes to the Prager-Hodge principle in the case of local plasticity (Prager and Hodge, 1951). It can be obtained from $\mathbf{\Omega}$ by imposing the equilibrium rate equation $(36)_1$, the rate elastic relation $(36)_3$ and the rate flow rule $(36)_4$. The relations $(36)_3$ and $(36)_4$ can be written, in terms of Fenchel's equality, according to $(30)_4$ and (23) so that we get the locally differentiable functional:

$$\mathbf{\Omega}_1(\dot{\boldsymbol{\sigma}},\dot{X}) = -\Psi^*(\dot{\boldsymbol{\sigma}},-\dot{X}) - \sqcup_{\mathcal{T}}(\dot{\boldsymbol{\sigma}},\dot{X}) + \Upsilon^*(\overline{\mathbf{B}}'\dot{\boldsymbol{\sigma}} - \dot{\ell}). \tag{38}$$

The following statement thus holds.

Proposition 6. *The pair* $(\dot{\boldsymbol{\sigma}},\dot{X})$ *is a solution of the optimization problem*:

$$\max_{\dot{\boldsymbol{\sigma}}}\ \operatorname*{stat}_{\dot{X}}\mathbf{\Omega}_1(\dot{\boldsymbol{\sigma}},\dot{X})$$

if and only if it is a solution of the nonlocal and gradient elastoplastic structural rate model (36).

Let us now derive a one-field potential which specializes to the Greenberg principle in the case of local plasticity (Greenberg, 1949). We enforce in the expression (37) of $\mathbf{\Omega}$ the compatibility rate condition $(36)_2$, the rate elastic relation $(36)_3$ in the form of Fenchel's equality $(30)_1$, the rate flow rule $(36)_4$ in terms of Fenchel's equality (23) and the rate external relation $(36)_5$ in terms of Fenchel's equality:

$$\Upsilon(u) + \Upsilon^*(r) = \langle r,u\rangle \tag{39}$$

to get the potential

$$\mathbf{\Omega}_2(\dot{u},\dot{\boldsymbol{\sigma}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa}) = \Psi(\overline{\mathbf{B}}\dot{u}-\dot{\boldsymbol{p}},\dot{\kappa}) - \sqcup_{\mathcal{T}}(\dot{\boldsymbol{\sigma}},\dot{X}) - \Upsilon(u) + ((\dot{\boldsymbol{\sigma}},\dot{\boldsymbol{p}})) + ((\dot{X},\dot{\kappa})) - \langle\dot{\ell},\dot{u}\rangle \tag{40}$$

which is convex in $(\dot{u},\dot{\boldsymbol{p}})$ and concave in $(\dot{\boldsymbol{\sigma}},\dot{X},\dot{\kappa})$.

We thus have

Proposition 7. *The set* $(\dot{u},\dot{\boldsymbol{\sigma}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa})$ *is a solution of the saddle problem*:

$$\min_{\dot{u},\dot{\boldsymbol{p}}}\ \max_{\dot{\boldsymbol{\sigma}},\dot{X},\dot{\kappa}}\mathbf{\Omega}_2(\dot{u},\dot{\boldsymbol{\sigma}},\dot{X},\dot{\boldsymbol{p}},\dot{\kappa})$$

if and only if it is a solution of the nonlocal and gradient elastoplastic structural rate model (36).

In order to eliminate the quartet $(\dot{\boldsymbol{\sigma}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa})$ from the independent variables of the potential $\boldsymbol{\Omega}_2$, we define the following convex functional:

$$W(\overline{\mathbf{B}}\dot{u}) = \inf_{\dot{\boldsymbol{p}}} \sup_{\dot{\boldsymbol{\sigma}}, \dot{X}, \dot{\kappa}} \{ \Psi(\overline{\mathbf{B}}\dot{u} - \dot{\boldsymbol{p}}, \dot{\kappa}) - \sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}) + ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\dot{X}, \dot{\kappa})) \}. \tag{41}$$

Substituting (41) in the potential $\boldsymbol{\Omega}_2$, we get the convex functional:

$$\boldsymbol{\Omega}_3(\dot{u}) = W(\overline{\mathbf{B}}\dot{u}) - \Upsilon(u) - \langle \dot{\ell}, \dot{u} \rangle \tag{42}$$

so that the next statement holds.

Proposition 8. *The displacement rate $\dot{u}$ is a solution of the convex optimization problem*:

$$\min_{\dot{u}} \boldsymbol{\Omega}_3(\dot{u})$$

if and only if it is a solution of the nonlocal and gradient elastoplastic structural rate model (36).

Let us now derive from the Propositions 6 and 8 the corresponding variational formulations in which the plastic multiplier explicitly appears as an independent variable.

The variational principle in $(\dot{\boldsymbol{\sigma}}, \lambda)$ has been introduced by Capurso (1969); Capurso and Maier (1970) for local plasticity. Its counterpart for nonlocal and gradient plasticity can be obtained from (38) by modifying the expression of $\sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X})$ according to the result provided in Appendix A:

$$\sqcup_{\mathscr{T}}(\dot{\boldsymbol{\sigma}}, \dot{X}, \lambda) = \begin{cases} ((\dot{\boldsymbol{\sigma}}, \lambda \, \mathrm{d}_{\boldsymbol{\sigma}} G(\boldsymbol{\sigma}, X))) - ((\dot{X}, \lambda)) - \sqcup_{\Re^+}(\lambda) & \text{if } G(\boldsymbol{\sigma}, X) = 0, \\ \sqcup_{\widehat{\mathscr{S} \times \mathscr{Y}'}}(\dot{\boldsymbol{\sigma}}, \dot{X}) & \text{if } G(\boldsymbol{\sigma}, X) < 0. \end{cases} \tag{43}$$

Taking into account the above expression (43), the expression (38) of the potential $\boldsymbol{\Omega}_1$ can be reworked to provide the next statement.

Proposition 9. *A triplet $(\dot{\boldsymbol{\sigma}}, \dot{X}, \lambda)$ is a solution of the optimization problem*:

$$\min_{\lambda} \max_{\dot{\boldsymbol{\sigma}}} \operatorname{stat}_{\dot{X}} \boldsymbol{\Omega}_4(\dot{\boldsymbol{\sigma}}, \dot{X}, \lambda),$$

where

$$\boldsymbol{\Omega}_4(\dot{\boldsymbol{\sigma}}, \dot{X}, \lambda) = -\Psi^*(\dot{\boldsymbol{\sigma}}, -\dot{X}) + \Upsilon^*(\overline{\mathbf{B}}'\dot{\boldsymbol{\sigma}} - \dot{\ell}) + \left\{ \begin{array}{ll} -((\dot{\boldsymbol{\sigma}}, \lambda \, \mathrm{d}_{\boldsymbol{\sigma}} G(\boldsymbol{\sigma}, X))) + ((\dot{X}, \lambda)) + \sqcup_{\Re^+}(\lambda) & \text{if } G(\boldsymbol{\sigma}, X) = 0, \\ -\sqcup_{\widehat{\mathscr{S} \times \mathscr{Y}'}}(\dot{\boldsymbol{\sigma}}, \dot{X}) & \text{if } G(\boldsymbol{\sigma}, X) < 0 \end{array} \right\}$$

if and only if it is a solution of the nonlocal and gradient elastoplastic structural rate model (36).

To get the nonlocal variational principle in $(\dot{u}, \lambda)$, which is the counterpart of the one introduced by Capurso (1969) and Capurso and Maier (1970) for local plasticity, we insert the expression (43) into the expression (41) of W in order to explicitly introduce the plastic multiplier λ:

$$W(\overline{\mathbf{B}}\dot{u}, \lambda) = \inf_{\dot{\boldsymbol{p}}} \sup_{\dot{\boldsymbol{\sigma}}, \dot{X}, \dot{\kappa}} \{ \Psi(\overline{\mathbf{B}}\dot{u} - \dot{\boldsymbol{p}}, \dot{\kappa}) + ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{p}})) + ((\dot{X}, \dot{\kappa})) + \left\{ \begin{array}{ll} -((\dot{\boldsymbol{\sigma}}, \lambda \, \mathrm{d}_{\boldsymbol{\sigma}} G(\boldsymbol{\sigma}, X))) + ((\dot{X}, \lambda)) + \sqcup_{\Re^+}(\lambda) & \text{if } G(\boldsymbol{\sigma}, X) = 0, \\ -\sqcup_{\widehat{\mathscr{S} \times \mathscr{Y}'}}(\dot{\boldsymbol{\sigma}}, \dot{X}) & \text{if } G(\boldsymbol{\sigma}, X) < 0. \end{array} \right\} \tag{44}$$

Hence it turns out to be

Proposition 10. *A pair* $(\dot{u}, \lambda)$ *is a solution of the convex optimization problem*:

$$\min_{\dot{u},\lambda} \mathbf{\Omega}_5(\dot{u}, \lambda),$$

where

$$\mathbf{\Omega}_5(\dot{u}, \lambda) = W(\bar{\mathbf{B}}\dot{u}, \lambda) - \Upsilon(\dot{u}) - \langle \dot{\ell}, \dot{u} \rangle \tag{45}$$

if and only if it is a solution of the nonlocal and gradient elastoplastic structural rate model (36).

5.2. Comparison with existing results

The nonlocal constitutive model and the related variational principles contributed in Borino et al. (1999) can be recovered as a special case of the treatment developed in this paper. In fact let us consider the following expression of the free energy

$$\Phi(e, \alpha_2) = \Phi_{\mathrm{el}}(e) + \Phi_{NL}(\xi(\alpha_2)) \tag{46}$$

which is convex in (e, α_2), and let the elastic domain be defined in the form:

$$C = \{(\sigma, \chi_2) \in \mathscr{S} \times \mathscr{Y}_2' : G(\sigma, \chi_2) = g(\sigma) - \chi_2 \leqslant 0\}. \tag{47}$$

The expressions (46) and (47) show that inelastic phenomena are governed by the nonlocal free energy Φ_{NL} and no local plastic behaviour is considered. Note that, in the notation adopted in Borino et al. (1999); the kinematic internal variable α_2 and the nonlocal static internal variable χ_2 are labelled κ and X respectively.

Remark 11. In Borino and Failla (2001), the constitutive model provided in Borino et al. (1999) is enhanced following the proposal made in Strömber and Ristinmaa (1996). Actually, in addition to the state variables (e, α_2) a local internal variable $\kappa \in \mathscr{Y}$ and its dual $X \in \mathscr{Y}'$ are introduced in order to account for isotropic softening. This model can be recovered from the model proposed in the present paper by considering the following expressions for the free energy and for the elastic domain:

$$\Phi(e, \kappa, \alpha_2) = \Phi_{\mathrm{el}}(e) + \Phi_L(\kappa) + \Phi_{NL}(\xi(\alpha_2)),$$
$$C = \{(\sigma, X, \chi_2) \in \mathscr{S} \times \mathscr{Y}' \times \mathscr{Y}_2' : G(\sigma, X, \chi_2) = g(\sigma) - X - \chi_2 \leqslant 0\}.$$

If the expressions (46) and (47) are considered, the variational formulation of the constitutive rate model reported in Proposition 4 becomes

Proposition 12. *For a given* $\dot{\varepsilon}$, *the plastic multiplier* λ *is a solution of the optimization problem*:

$$\mathrm{stat}\{\mathbf{P}(\lambda) \text{ s.t. } \lambda \geqslant 0,\ G(\sigma(\lambda), \chi_2(\lambda)) \leqslant 0,\ \lambda G(\sigma(\lambda), \chi_2(\lambda)) = 0\},$$

where

$$\mathbf{P}(\lambda) = \tfrac{1}{2}((\mathbf{E}\lambda\, \mathrm{d}g(\sigma), \lambda\, \mathrm{d}g(\sigma))) - ((\mathbf{E}\dot{\varepsilon}, \lambda\, \mathrm{d}g(\sigma))) + \tfrac{1}{2}((\mathsf{H}_2\lambda, \lambda))$$

if and only if it is a solution of the nonlocal elastoplastic rate model proposed in Borino et al. (1999).

The expression of the potential $\mathbf{P}(\lambda)$ coincides with the expression of $\Pi[\lambda]$ reported in Borino et al. (1999) by noting the equality $((\mathsf{H}_2\lambda, \lambda)) = ((\mathbf{R}'\,\mathrm{d}^2\Phi_{NL}(\xi)\mathbf{R}\lambda, \lambda)) = ((\mathrm{d}^2\Phi_{NL}(\xi)\mathbf{R}\lambda, \mathbf{R}\lambda))$. It is worth noting that the nonnegativeness condition of the quadratic term $\mathsf{H}_2\lambda \cdot \lambda + \mathbf{E}\lambda \mathrm{d}g(\sigma) \cdot \lambda \mathrm{d}g(\sigma)$, which has been introduced in the statement of the analogous variational formulation reported in the quoted paper, is inessential as shown in Proposition 12 above where the minimum set of essential conditions is reported.

Assuming bilateral frictionless external constraints, the Proposition 7 involving the potential $\mathbf{\Omega}_2$ can be explicitly rewritten in the form:

Proposition 13. *The set* $(\dot{u}, \dot{\sigma}, \dot{\chi}_2, \dot{p}, \dot{\alpha}_2)$ *is a solution of the saddle problem*:

$$\min_{\dot{u},\dot{p}} \max_{\dot{\sigma},\dot{\chi}_2,\dot{\alpha}_2} \mathbf{L}_1(\dot{u}, \dot{\sigma}, \dot{\chi}_2, \dot{p}, \dot{\alpha}_2)$$

subject to the conditions $\dot{u} \in \mathscr{L}$ *and* $((\dot{\sigma}, \dot{p})) - ((\dot{\chi}_2, \dot{\alpha}_2)) \leqslant 0$ *where*

$$\mathbf{L}_1(\dot{u}, \dot{\sigma}, \dot{\chi}_2, \dot{p}, \dot{\alpha}_2) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\dot{u} - \dot{p}), \mathbf{B}\dot{u} - \dot{p})) + \tfrac{1}{2}((\mathsf{H}_2\dot{\alpha}_2, \dot{\alpha}_2)) + ((\dot{\sigma}, \dot{p})) - ((\dot{\chi}_2, \dot{\alpha}_2)) - \langle \dot{\ell}, \dot{u} \rangle \tag{48}$$

if and only if it is a solution of the nonlocal elastoplastic structural rate model proposed in Borino et al. (1999).

In Borino et al. (1999) the expressions $((\mathsf{H}_2\dot{\alpha}_2, \dot{\alpha}_2))$ and $((\dot{\chi}_2, \dot{\alpha}_2))$, appearing in $\mathbf{L}_1$, are provided in terms of the nonlocal variable ξ and of its dual local variable χ. Actually, recalling (2) and the equality $\mathsf{H}_2 = \mathbf{R}' d^2\Phi_{NL}(\xi)\mathbf{R}$, we have

$$((\mathsf{H}_2\dot{\alpha}_2, \dot{\alpha}_2)) = ((\mathbf{R}'\,\mathrm{d}^2\Phi_{NL}(\xi)\mathbf{R}\dot{\alpha}_2, \dot{\alpha}_2)) = ((\mathrm{d}^2\Phi_{NL}(\xi)\dot{\xi}, \dot{\xi})),$$

$$((\dot{\chi}_2, \dot{\alpha}_2)) = ((\mathbf{R}'\dot{\chi}, \dot{\alpha}_2)) = ((\dot{\chi}, \dot{\xi}))$$

so that the Proposition 13 coincides with the corresponding one reported in Borino et al. (1999).

The kinematic-type variational principle reported in Borino et al. (1999) can be obtained from Proposition 10. In fact evaluating the expression (44) it turns out to be

$$W(\overline{\mathbf{B}}\dot{u}, \lambda) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma)), \mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma))) + \tfrac{1}{2}((\mathsf{H}_2\lambda, \lambda)) \tag{49}$$

subject to $\lambda \geqslant 0$ if $G(\sigma,X) = 0$ or $\lambda = 0$ if $G(\sigma,X) < 0$. Hence we have:

Proposition 14. *A pair* $(\dot{u}, \lambda)$ *is a solution of the concave optimization problem*:

$$\min_{\dot{u},\lambda} \mathbf{L}_2(\dot{u}, \lambda)$$

under the conditions $\dot{u} \in \mathscr{L}$, $\lambda \geqslant 0$ *if* $G(\sigma,X) = 0$ *or* $\lambda = 0$ *if* $G(\sigma,X) < 0$, *where*

$$\mathbf{L}_2(\dot{u}, \lambda) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma)), \mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma))) + \tfrac{1}{2}((\mathrm{d}^2\Phi_{NL}(\xi)\mathbf{R}\lambda, \mathbf{R}\lambda)) - \langle \dot{\ell}, \dot{u} \rangle \tag{50}$$

if and only if it is a solution of the nonlocal elastoplastic structural rate model proposed in Borino et al. (1999).

Let us now show that Proposition 10 can be specialized to an analogous principle proposed by Mühlhaus and Aifantis (1991) for gradient plasticity, see also e.g. de Borst and Mühlhaus (1992); Fleck and Hutchinson (2001). To this end the generalized vectors are defined by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}, \qquad \boldsymbol{e} = \begin{bmatrix} e \\ \alpha_2 \end{bmatrix}, \qquad \boldsymbol{p} = \begin{bmatrix} p \\ -\alpha_2 \end{bmatrix}, \qquad \boldsymbol{\sigma} = \begin{bmatrix} \sigma \\ \chi_2 \end{bmatrix} \tag{51}$$

so that the expression (44) turns out to be:

$$W(\overline{\mathbf{B}}\dot{u}, \lambda) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma)), \mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma))) + \tfrac{1}{2}((\mathsf{H}_2\lambda, \lambda)) + \tfrac{1}{2}((\mathbf{H}\lambda, \lambda)) \tag{52}$$

subject to $\lambda \geqslant 0$ if $G(\sigma,X) = 0$ or $\lambda = 0$ if $G(\sigma,X) < 0$. Noting that $((\mathsf{H}_2\lambda, \lambda)) = ((c\,\mathrm{d}^2\Phi_{NL}(\xi)\nabla\lambda, c\nabla\lambda))$ we have

Proposition 15. *A pair* $(\dot{u}, \lambda)$ *is a solution of the convex optimization problem*:

$$\min_{\dot{u},\lambda} \mathbf{G}(\dot{u}, \lambda)$$

under the conditions $\dot{u} \in \mathscr{L}$, $\lambda \geqslant 0$ *if* $G(\sigma, X) = 0$ *or* $\lambda = 0$ *if* $G(\sigma, X) < 0$, *where*

$$\mathbf{G}(\dot{u}, \lambda) = \tfrac{1}{2}((\mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)), \mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma))) + \tfrac{1}{2}((\mathsf{H}_2\lambda, \lambda)) + \tfrac{1}{2}((\mathbf{H}\lambda, \lambda)) - \langle \dot{\ell}, \dot{u} \rangle \tag{53}$$

if and only if it is a solution of the gradient elastoplastic structural rate model.

In order to explicitly prove that the stationarity of $\mathbf{G}$ is equivalent to the rate gradient model, we note that the stationarity of $\mathbf{G}$ is

$$(0,0) \in \partial \mathbf{G}(\dot{u}, \lambda) \iff \begin{cases} 0 \in \partial_{\dot{u}} \mathbf{G}(\dot{u}, \lambda) \iff \mathbf{B}'\mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)) = \dot{\ell}, \\ 0 \in \partial_{\lambda} \mathbf{G}(\dot{u}, \lambda) \iff 0 \in -[\mathrm{d}g(\sigma)]'\mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)) + \mathsf{H}_2\lambda + \mathbf{H}\lambda + \partial \sqcup_{\Re^+}(\lambda). \end{cases} \tag{54}$$

The relation $(54)_1$ provides the stress rate $\dot{\sigma} = \mathbf{E}(\mathbf{B}\dot{u} - \lambda \mathrm{d}g(\sigma))$ which turns out to be in equilibrium with the external load rate $\dot{\ell}$, i.e. $\mathbf{B}'\dot{\sigma} = \dot{\ell}$. The relation $(54)_2$ yields Prager's consistency condition:

$$((\dot{\sigma}, \lambda\, \mathrm{d}g(\sigma))) - ((\dot{\chi}_2, \lambda)) - ((\dot{X}, \lambda)) = 0, \tag{55}$$

where $\dot{\chi}_2 = \mathsf{H}_2\lambda$, $\dot{X} = \mathbf{H}\lambda$ and the conditions $\lambda \geqslant 0$ and $G(\dot{\sigma}, \dot{X}) = [\mathrm{d}g(\sigma)]'\mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)) - \mathsf{H}_2\lambda - \mathbf{H}\lambda \leqslant 0$.

The functional $\mathbf{G}$ can be rewritten for the Cauchy model in the following form:

$$\mathbf{G}(\dot{u}, \lambda) = \frac{1}{2}\int_\Omega \mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)) \cdot (\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma))\, \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega c^2\, \mathrm{d}\chi(\xi) \nabla\lambda \cdot \nabla\lambda\, \mathrm{d}\mathbf{x} + \frac{1}{2}\int_\Omega \mathbf{H}\lambda^2\, \mathrm{d}\mathbf{x} - \langle \dot{\ell}, \dot{u} \rangle$$

and the expression (55)

$$\int_\Omega \mathbf{E}(\mathbf{B}\dot{u} - \lambda\, \mathrm{d}g(\sigma)) \cdot \lambda\, \mathrm{d}g(\sigma)\, \mathrm{d}\mathbf{x} - \int_\Omega c^2\, \mathrm{d}\chi(\xi) \nabla\lambda \cdot \nabla\lambda\, \mathrm{d}\mathbf{x} - \int_\Omega \mathbf{H}\lambda \cdot \lambda\, \mathrm{d}\mathbf{x} = 0 \tag{56}$$

provides the Prager's consistency condition.

6. Conclusion

A nonlocal and gradient model of plasticity is presented and is cast in the framework of convex analysis and of the potential theory for monotone multivalued operators. As a consequence a theoretical analysis can be performed in analogy with local standard plasticity and variational formulations for the rate constitutive model are contributed. The rate nonlocal and gradient structural problem is then formulated and the related variational formulations are provided. It is shown that nonlocal and gradient models and related variational formulations, recently contributed in the literature, can be recovered as a special case of the present model. The proposed treatment of plasticity is rather general and can be applied to further different material behaviours which can be described within the theory of internal variables such as damage and rate-dependent plasticity.

A discussion of approximation methods and of finite-step nonlocal and gradient plasticity deserves further analysis and will be the subject of a forthcoming paper.

Acknowledgements

The financial support of the Italian Ministry for Instruction, University and Research is gratefully acknowledged.

Appendix A

For simplicity we collect local and nonlocal variables in the vectors $\underset{\smile}{\boldsymbol{\varepsilon}} = [\,\boldsymbol{\varepsilon} \quad 0\,]^{\mathrm{T}}$, $\underset{\smile}{\boldsymbol{e}} = [\,\boldsymbol{e} \quad \kappa\,]^{\mathrm{T}}$, $\underset{\smile}{\boldsymbol{p}} = [\,\boldsymbol{p} \quad \kappa\,]^{\mathrm{T}}$, $\underset{\smile}{\boldsymbol{\sigma}} = [\,\boldsymbol{\sigma} \quad X\,]^{\mathrm{T}}$ and we define the dual spaces $\underset{\smile}{\mathscr{D}} = \widehat{\mathscr{D}} \times \mathscr{Y}$ and $\underset{\smile}{\mathscr{S}} = \widehat{\mathscr{S}} \times \mathscr{Y}'$. The relevant scalar product is denoted by the symbol $\prec \bullet, \bullet \succ$ defined as $\prec \underset{\smile}{\boldsymbol{\sigma}}, \underset{\smile}{\boldsymbol{e}} \succ = ((\boldsymbol{\sigma}, \boldsymbol{e})) + ((X, \kappa)) = ((\sigma, e)) + ((\chi_1, \alpha_1)) + ((\chi_2, \alpha_2)) + ((X, \kappa))$.

- Let us prove the equality $\sqcup_{\mathscr{N}}(\dot{\underset{\smile}{\boldsymbol{p}}}) = \sqcup^*_{\mathscr{T}}(\dot{\underset{\smile}{\boldsymbol{p}}})$. By definition of support functional we have

  $$\sqcup^*_{\mathscr{T}}(\dot{\underset{\smile}{\boldsymbol{p}}}) = \sup\{\prec \underset{\smile}{\boldsymbol{\tau}}, \dot{\underset{\smile}{\boldsymbol{p}}} \succ \,|\, \underset{\smile}{\boldsymbol{\tau}} \in \mathscr{T}\}.$$

  If $\dot{\underset{\smile}{\boldsymbol{p}}} \in \mathscr{N}$ we have $\prec \underset{\smile}{\boldsymbol{\tau}}, \dot{\underset{\smile}{\boldsymbol{p}}} \succ \leqslant 0$ for any $\underset{\smile}{\boldsymbol{\tau}} \in \mathscr{T}$ and if $\dot{\underset{\smile}{\boldsymbol{p}}} \notin \mathscr{N}$, there exists a $\underset{\smile}{\boldsymbol{\tau}} \in \mathscr{T}$ such that $\prec \underset{\smile}{\boldsymbol{\tau}}, \dot{\underset{\smile}{\boldsymbol{p}}} \succ > 0$. Accordingly we have the result

  $$\sqcup^*_{\mathscr{T}}(\dot{\underset{\smile}{\boldsymbol{p}}}) = \begin{cases} 0 & \text{if } \dot{\underset{\smile}{\boldsymbol{p}}} \in \mathscr{N}, \\ +\infty & \text{if } \dot{\underset{\smile}{\boldsymbol{p}}} \notin \mathscr{N}, \end{cases} = \sqcup_{\mathscr{N}}(\dot{\underset{\smile}{\boldsymbol{p}}}).$$

- Let us now prove the expression (43) for the tangent cone to the elastic domain C at the point $\underset{\smile}{\boldsymbol{\sigma}}$. The tangent cone to the elastic domain C at the point $\underset{\smile}{\boldsymbol{\sigma}}$ can be written in the following form:

  $$\mathscr{T} = \begin{cases} \{\dot{\underset{\smile}{\boldsymbol{\sigma}}} \in \underset{\smile}{\mathscr{S}} \mid \dot{G}(\underset{\smile}{\boldsymbol{\sigma}}) = \prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ \leqslant 0 & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) = 0, \\ \underset{\smile}{\mathscr{S}} & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) < 0 \end{cases}$$

so that the indicator of the cone $\mathscr{T}$ becomes

$$\sqcup_{\mathscr{T}}(\dot{\underset{\smile}{\boldsymbol{\sigma}}}) = \begin{cases} \sqcup_{\Re^-}[\prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ] & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) = 0 \\ \sqcup_{\underset{\smile}{\mathscr{S}}}(\dot{\underset{\smile}{\boldsymbol{\sigma}}}) & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) < 0. \end{cases} \tag{A.1}$$

A pair $(\dot{\underset{\smile}{\boldsymbol{p}}}, \dot{\underset{\smile}{\boldsymbol{\sigma}}})$, which satisfies the rate flow rule (24), fulfils one of the following two relations:

$$\begin{aligned} &\text{(i) } \dot{\underset{\smile}{\boldsymbol{p}}} \in \partial \sqcup_{\Re^-} [\prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ] \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) = \lambda \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \quad \text{with} \quad \lambda \in \partial \sqcup_{\Re^-} [\prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ] \quad \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) = 0, \\ &\text{(ii) } \dot{\underset{\smile}{\boldsymbol{p}}} \in \partial \sqcup_{\underset{\smile}{\mathscr{S}}} (\dot{\underset{\smile}{\boldsymbol{\sigma}}}) \qquad \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) < 0. \end{aligned} \tag{A.2}$$

By means of Fenchel's equality, the constraint condition $\lambda \in \partial \sqcup_{\Re^-} [\mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \dot{\underset{\smile}{\boldsymbol{\sigma}}}]$ can be equivalently written in the form $\sqcup_{\Re^-}[\prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ] + \sqcup_{\Re^+}(\lambda) = \prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \lambda \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ$. Accordingly the expression (A.1) of the indicator of $\mathscr{T}$ becomes

$$\sqcup_{\mathscr{T}}(\dot{\underset{\smile}{\boldsymbol{\sigma}}}, \lambda) = \begin{cases} \prec \dot{\underset{\smile}{\boldsymbol{\sigma}}}, \lambda \mathrm{d}G(\underset{\smile}{\boldsymbol{\sigma}}) \succ - \sqcup_{\Re^+}(\lambda) & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) = 0, \\ \sqcup_{\underset{\smile}{\mathscr{S}}}(\dot{\underset{\smile}{\boldsymbol{\sigma}}}) & \text{if } G(\underset{\smile}{\boldsymbol{\sigma}}) < 0. \end{cases}$$

Appendix B

Let us now prove Proposition 2. To this end the rate flow rule $(27)_2$ is inverted by considering the equivalent expression $(24)_2$ and the rate elastic relation $(27)_3$ is rewritten in the equivalent form $(29)_4$. Recalling that $\sqcup_{\mathscr{N}} = \sqcup^*_{\mathscr{T}}$, the operator form of the constitutive relations (27) is given by

$$\begin{bmatrix} \circ \\ \circ \\ 0 \\ \circ \\ 0 \end{bmatrix} \in \mathbf{C} \begin{bmatrix} \dot{\boldsymbol{\sigma}} \\ \dot{\boldsymbol{e}} \\ \dot{X} \\ \dot{\boldsymbol{p}} \\ \dot{\kappa} \end{bmatrix} + \begin{bmatrix} \dot{\boldsymbol{\varepsilon}} \\ \circ \\ 0 \\ \circ \\ 0 \end{bmatrix}$$

with

$$\mathbf{C} = \begin{bmatrix} 0 & -I_{\widehat{\mathscr{D}}} & 0 & -I_{\widehat{\mathscr{D}}} & 0 \\ -I_{\widehat{\mathscr{S}}} & & & 0 & 0 \\ & \multicolumn{2}{c}{\mathrm{d}\Xi^*} & & \\ 0 & & & 0 & -I_{\mathscr{Y}} \\ -I_{\widehat{\mathscr{S}}} & 0 & 0 & & \\ & & & \multicolumn{2}{c}{\partial \sqcup_{\mathscr{N}}} \\ 0 & 0 & -I_{\mathscr{Y}'} & & \end{bmatrix},$$

where $\mathbf{C}$ is the multivalued *constitutive operator*. The operator $\mathbf{C}$ can be split in a linear symmetric operator, and hence conservative (Vainberg, 1964), and in the multivalued operator $\partial \sqcup_{\mathscr{N}}$ which is conservative by virtue of the integral theorem (Romano et al., 1993).

The potential $\boldsymbol{\Sigma}$ can then be evaluated by a direct integration along a straight line in the space $\widehat{\mathscr{S}} \times \widehat{\mathscr{D}} \times \mathscr{Y}' \times \widehat{\mathscr{D}} \times \mathscr{Y}$ to get

$$\boldsymbol{\Sigma}(\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}}, \dot{X}, \dot{\boldsymbol{p}}, \dot{\kappa}) = -\int_0^1 ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{e}} + \dot{\boldsymbol{p}}))\mathrm{d}t + \int_0^1 ((\mathrm{d}\Xi^*(\dot{\boldsymbol{e}}, \dot{X}), (\dot{\boldsymbol{e}}, \dot{X})))\mathrm{d}t - \int_0^1 ((\dot{X}, \dot{\kappa}))\,\mathrm{d}t + \sqcup_{\mathscr{N}}(\dot{\boldsymbol{p}}, \dot{\kappa}) + ((\dot{\boldsymbol{\sigma}}, \dot{\boldsymbol{\varepsilon}}))$$

and the expression (31) is obtained.

Appendix C

Some basic elements of convex analysis which have been referred to in the paper are reported. For more details see e.g. Hiriart-Urruty and Lemarechal (1993).

Let $\mathscr{X}$ and $\mathscr{X}'$ be a pair of locally convex topological vector spaces placed in separating duality by a bilinear form $\langle \bullet, \bullet \rangle$.

The conjugate of a convex functional $f : \mathscr{X} \to \Re \cup \{+\infty\}$ is the convex closed functional $f^* : \mathscr{X}' \to \Re \cup \{+\infty\}$ defined by

$$f^*(x^*) = \sup_{y \in \mathscr{X}} \{\langle x^*, y \rangle - f(y)\}.$$

If f is closed we have $f^{**} = f$.

Given a set $K \subset \mathscr{X}$, the indicator of K at a point $x \in \mathscr{X}$ is defined as follows:

$$\sqcup_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{otherwise.} \end{cases}$$

The normal cone to a convex set K at a point x is

$$N_K(x) = \begin{cases} \{x^* \in \mathscr{X}' : \langle x^*, y - x\rangle \leqslant 0 \quad \forall y \in \mathscr{X}\} & \text{if } x \in K, \\ \emptyset & \text{otherwise.} \end{cases}$$

The tangent cone to a convex set K at a point x is given by

$$T_K(x) = \{y \in \mathscr{X} : \langle x^*, y\rangle \leqslant 0 \quad \forall x^* \in N_K(x)\}.$$

If x belong to the interior of K, the tangent cone coincides with the whole space $\mathscr{X}$ and y is arbitrary. On the contrary if x belongs to the boundary of K, the tangent cone turns out to be a proper subset of $\mathscr{X}$.

Given a set $K \subseteq \mathscr{X}$, the support functional of K at a point $x^* \in \mathscr{X}'$ is defined as:

$$D(x^*) = \sup_{y \in K} \langle x^*, y\rangle.$$

The subdifferential of a convex functional $f : \mathscr{X} \to \Re \cup \{+\infty\}$, having a nonempty domain, is the set $\partial f(x) \subseteq \mathscr{X}'$ such that:

$$x^* \in \partial f(x) \Longleftrightarrow f(y) - f(x) \geqslant \langle x^*, y - x\rangle \quad \forall y \in \mathscr{X}.$$

In particular, if the functional f is differentiable at x, the subdifferential coincides with the differential.

Given a closed convex conjugate functional f and its conjugate f^*, the following relations are equivalent:

$$x^* \in \partial f(x) \quad x \in \partial f^*(x^*) \quad f(x) + f^*(x^*) = \langle x^*, x\rangle,$$

where the last relation is known as the Fenchel's equality.

The subdifferential of the indicator functional of a convex set K at a point $x \in K$ coincides with the normal cone to K at x, i.e. $\partial \sqcup_K (x) = N_K(x)$. The support and the indicator functionals of a convex set K are conjugate so that the following relations hold:

$$x^* \in \partial \sqcup_K (x) = N_K(x) \quad x \in \partial D(x^*) \quad \sqcup_K (x) + D(x^*) = \langle x^*, x\rangle.$$

Given a monotone convex function $m : \Re \to \Re \cup \{+\infty\}$ and a continuous convex functional $f : \mathscr{X} \to \Re \cup \{+\infty\}$, the functional (mf) is convex and its subdifferential at a point $x \in \mathscr{X}$, which is not a minimum for f, is given by

$$\partial(m \circ f)(x) = \partial m[f(x)]\partial f(x).$$

Analogous results hold for concave functionals.

References

Aifantis, E.C., 1987. The physics of plastic deformation. Int. J. Eng. Plasticity 3, 211–247.

Bažant, Z.P., Belytschko, T.B., Chang, T.P., 1984. Continuum theory for strain-softening. J. Eng. Mech. (ASCE) 110, 1666–1692.

Bažant, Z.P., Pijaudier-Cabot, G., 1988. Nonlocal continuum damage, localization instability and convergence. J. Appl. Mech. (ASME) 55, 287–293.

Borino, G., Fuschi, P., Polizzotto, C., 1999. A thermodynamic approach to nonlocal plasticity and related variational principles. J. Appl. Mech. (ASME) 66, 952–963.

Borino, G., Failla, B., 2001. Thermodynamic consistent plasticity models with local and nonlocal internal variables. In: European Congress an Computational Methods in Applied Sciences and Engineering, CD-Rom Proceedings.

Brezis, H., 1983. Analyse Fonctionnelle, Theorie et applications. Masson Editeur, Paris.

Capurso, M., 1969. Principi di minimo per la soluzione incrementale dei problemi elastoplastici. Parte I e II. Rend. Acc. Naz. Lincei XLVI (April–May).

Capurso, M., Maier, G., 1970. Incremental elastoplastic analysis and quadratic optimization. Meccanica V, 107–117.

Coleman, B.D., Hodgdon, M.L., 1985. On shear bands in ductile materials. Arch. Rat. Mech. Anal. 90, 219–249.

Cosserat, E., Cosserat, F., 1909. Théorie des Corps Déformables. Hermann, Paris.

de Borst, R., Mühlhaus, H.B., 1992. Gradient-dependent plasticity: formulation and algorithmic aspects. Int. J. Numer. Meth. Eng. 35, 521–539.

de Borst, R., Sluys, L.J., Mühlhaus, H.B., Pamin, J., 1993. Fundamental issues in finite element analysis of localization of deformation. Eng. Comput. 10, 99–122.

de Borst, R., Pamin, J., Sluys, L.J., 1995. Gradient plasticity for localization problems in quasi-brittle and frictional materials. in: Owen E. Onate, Hinton, E. (Eds.), Computational Plasticity, Fundamentals and Applications, DRJ, vol. 1. Pineridge Press, Swansea, UK, pp. 509–533.

de Borst, R., 2001. Some recent issues in computational failure mechanics. Int. J. Numer. Meth. Eng. 52, 63–95.

Dillon, O.W., Kratochvil, J., 1970. A strain gradient theory of plasticity. Int. J. Solids Struct. 6, 1513–1533.

Ericksen, J.L., Truesdell, C., 1958. Exact theory of stress and strain in rods and shells. Arch. Rational Mech. Anal. 1, 295–323.

Eringen, A.C., 1966. A unified theory of thermomechanical materials. Int. J. Eng. Sci. 4, 179–202.

Eringen, A.C., Edelen, D.G.B, 1972. On nonlocal elasticity. Int. J. Eng. Sci. 10, 233–248.

Fleck, N.A., Hutchinson, J.W., 2001. A reformulation of strain gradient plasticity. J. Mech. Phys. Solids 49, 2245–2271.

Greenberg, H.J., 1949. Complementary minimum principles for an elastic–plastic material. Quart. Appl. Math. 7, 85.

Halphen, B., Nguyen, Q.S., 1975. Sur les materiaux standards generalises. J. Mech. 14, 39–63.

Hill, R., 1950. The Mathematical Theory of Plasticity. Clarendon Press, Oxford, UK.

Hiriart-Urruty, J.B., Lemarechal, C., 1993. Convex Analysis and Minimization Algorithms I. Springer-Verlag, New York.

Lasry, D., Belytschko, T., 1988. Localization limiters and transient problems. Int. J. Solids Struct. 24, 581–597.

Lubliner, J., 1990. Plasticity theory. Macmillan Publishing C, New York.

Mühlhaus, H.B., Aifantis, E.C., 1991. A variational principle for gradient plasticity. Int. J. Solids Struct. 28, 845–857.

Needleman, A., 1988. Material rate dependence and mesh sensitivity in localization problems. Comp. Meth. Appl. Mech. Eng. 67, 69–86.

Nguyen, Q.S., 1977. On the elastic plastic initial-boundary value problem and its numerical integration. Int. J. Numer. Meth. Eng. 11, 817–832.

Pijaudier-Cabot, G., Bažant, Z.P., 1987. Nonlocal damage theory. J. Eng. Mech. (ASCE) 113, 127–144.

Pijaudier-Cabot, G., Benallal, A., 1993. Strain localization and bifurcation in a nonlocal continuum. Int. J. Solids Struct. 30, 1761–1775.

Prager, W., Hodge, P.G., 1951. Theory of Perfect Plastic Solids. John Wiley, New York.

Reddy, B.D., Martin, J.B., 1991. Algorithms for the solution of internal variable problems in plasticity. Comp. Meth. Appl. Mech. Eng. 93, 253–273.

Romano, G., 2002. Structural Mechanics, vol. 2. Hevelius, Naples (in Italian).

Romano, G., 1995. New results in subdifferential calculus with applications to convex optimization. Appl. Math. Optim. 32, 213–234.

Romano, G., Rosati, L., Marotti de Sciarra, F., 1992. An internal variable theory of inelastic behaviour derived from the uniaxial rigid-perfectly plastic law. Int. J. Eng. Sci. 31, 1105–1120.

Romano, G., Rosati, L., Marotti de Sciarra, F., Bisegna, P., 1993. A potential theory for monotone multi-valued operators. Quart. Appl. Math. LI, 613–631.

Salençon, J., 1983. Calcul à la Rupture et Analyse Limite. Presses Ponts et Chaussées, Paris.

Simo, J.C., 1988. A framework for finite strain elastoplasticity based on maximum plastic dissipation and the multiplicative decomposition. Part I - Continuum formulation. Comp. Meth. Appl. Mech. Eng. 66, 199–219.

Simo, J.C., Kennedy, J., Govindjee, S., 1988. Non-smooth multisurface plasticity and viscoplasticity. Loading/unloading conditions and numerical algorithms. Int. J. Numer. Meth. Eng. 26, 2161–2185.

Sluys, L.J., de Borst, R., 1992. Wave propagation and localization in rate dependent crack medium, model formulation and one dimensional examples. Int. J. Solids Struct. 29, 2945–2958.

Strömber, L., Ristinmaa, M., 1996. FE-formulation of a nonlocal plasticity theory. Comp. Meth. Appl. Mech. Eng. 136, 127–144.

Svedberg, T., Runesson, K., 1998. Thermodynamically consistent nonlocal and gradient formulations of plasticity, EUROMECH Colloqium 378, Mulhouse, France.

Vainberg, M.M., 1964. Variational Methods for the Study of Nonlinear Operators. Holden-Day, Inc, San Francisco.